NORM CONVOLUTION INEQUALITIES IN LEBESGUE SPACES

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Abstract. We obtain upper and lower estimates of the \((p, q)\) norm of the convolution operator. The upper estimate sharpens the Young-type inequalities due to O’Neil and Stepanov.

1. Introduction

Let \(1 \leq p \leq \infty\), \(L_p \equiv L_p(\mathbb{R})\) and let the convolution operator be given by

\[(Af)(x) = (K \ast f)(x) = \int_{\mathbb{R}} K(x - y) f(y) dy.\]  

(1.1)

The Young convolution inequality

\[\|A\|_{L^p \to L^q} \leq \|K\|_{L^r}, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \quad 1 \leq p \leq q \leq \infty,\]

plays a very important role both in Harmonic Analysis and PDE theory. We note however that this estimate does not allow us to deal with power kernels such as \(K(x) = |x|^{-\gamma}, \gamma > 0\).

Young’s estimates were generalized by O’Neil \([ON]\) who showed that for \(1 < p < q < \infty\) and \(1/r = 1 - 1/p + 1/q\)

\[\|A\|_{L^p \to L^q} \leq C \|K\|_{L^{r, \infty}} := C \sup_{t>0} t^{1/r} K^*(t),\]

(1.2)

where \(K^*(t) = \inf \{\sigma : \mu \{x \in \Omega : |f(x)| > \sigma\} \leq t\}\) is the decreasing rearrangement of \(K\). In particular, this gives the Hardy-Littlewood fractional integration theorem, which corresponds to the model case of convolution by \(K(x) = |x|^{-1/r}\).

Another extension of Young’s inequality was proved by Stepanov \([Stp]\) using the Wiener amalgam space \(W(L_{r, \infty}[0, 1], L_{r, \infty}(\mathbb{Z}))\) (see e.g. \([Fe]\)): for \(1 < p < q < +\infty\) and \(1/r = 1 - 1/p + 1/q\) one has

\[\|A\|_{L^p \to L^q} \leq C \|K\|_{W(L_{r, \infty}[0, 1], L_{r, \infty}(\mathbb{Z}))},\]

(1.3)

2000 Mathematics Subject Classification. Primary 46E30; Secondary 44A35, 47G10.

Key words and phrases. Convolution, Young-O’Neil inequality, Lorentz spaces.
where
\[
\|K\|_{W(L_r,[0,1],L_r(Z))} := \left\|\tilde{K}\right\|_{L_r([0,1])} := \sup_{n \in \mathbb{N}} n^{1/r} \left(\sup_{0 \leq t \leq 1} t^{1/r} \tilde{K}^*(t,\cdot)\right)^*_n,
\]
and \(\tilde{K}(x,m) := K(m+x), \quad m \in \mathbb{Z}, \quad x \in [0,1]\). In [Stp] it was also shown that inequalities (1.2) and (1.3) are not comparable.

The aim of the present paper is to give upper and lower estimates of \(\|A\|_{L_p \to L_q}\) so that the upper estimate improves both (1.2) and (1.3). To formulate our main results, we will need the following definitions.

Let \(I\) be an interval with \(|I| = d\). Then \(T_I = \{I + kd\}_{k \in \mathbb{Z}}\) is a partition of \(\mathbb{R}\), i.e., \(\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (I + kd)\). We define two collections of sets \(\mathcal{L}(I) \subset \mathcal{U}(I)\):

\[
\mathcal{L}(I) = \left\{ e : e = \bigcup_{k=1}^m (\lbrack a,b \rbrack + kd), \quad [a,b] \subseteq I, \quad m \in \mathbb{N} \right\}
\]
and
\[
\mathcal{U}(I) = \left\{ e : e = \bigcup_{k=1}^m \omega_k, \quad m \in \mathbb{N} \right\},
\]
where \(\{\omega_k\}_{m}^1\) is any collection of compact sets of equal measure \(|\omega_k| \leq d\) and such that each \(\omega_k\) belongs to a different elements of \(T_I\).

**Theorem.** Let \(1 < p < q < \infty\) and \(K \in L_{loc}\). Then for \(Af = K * f\) we have

\[
C_1 \sup_I \sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/p - 1/q}} \left| \int_e K(x)dx \right| \leq \|A\|_{L_p \to L_q} \leq C_2 \inf_I \sup_{e \in \mathcal{U}(I)} \frac{1}{|e|^{1/p - 1/q}} \left| \int_e K(x)dx \right|,
\]
where the constants \(C_1\) and \(C_2\) depend only on \(p\) and \(q\).

For the certain regular kernels \(K\), for instance, monotone or quasi-monotone, the upper and lower bounds in (1.5) coincide, that is, we get the equivalent relation for \(\|A\|_{L_p \to L_q}\). More precisely, we call a locally integrable function \(K(x)\) weakly monotone if there exists a constant \(C > 0\) such that for any \(x \in \mathbb{R} \setminus \{0\}\)

\[
|K(x)| \leq C \left| \frac{1}{x} \int_0^x K(t)dt \right|.
\]

**Corollary.** Let \(1 < p < q < \infty\) and \(K \in L_{loc}\) be a weakly monotone function. Then a necessary and sufficient condition for \(Af = K * f\) to be bounded from \(L_p(\mathbb{R})\) to \(L_q(\mathbb{R})\) is

\[
\sup_{|x| > 0} \frac{1}{|x|^{1/p - 1/q}} \left| \int_0^x K(y)dy \right| < \infty.
\]
Moreover,
\[ \|A\|_{L_p \rightarrow L_q} \approx \sup_{|x|>0} \frac{1}{|x|^{1/p-1/q}} \left| \int_0^x K(y)dy \right|. \]

By \( C, C', c \) we will denote positive constants that may be different on different occasions. We write \( F \approx G \) if \( F \leq C_1 G \) and \( G \leq C_2 F \) for some positive constants \( C_1 \) and \( C_2 \) independent of essential quantities involved in the expressions \( F \) and \( G \).

The paper is organized as follows. In section 2 we obtain a required version of the Riesz Lemma for rearrangements (see, e.g., [St]). Section 3 and 4 are devoted to the estimates of \( \|A\|_{L_p \rightarrow L_q} \) from above and below, correspondingly. We conclude with Section 5, where we show that the right-hand side estimate in (1.5) implies both (1.2) and (1.3) but the reverse does not hold in general.

2. Rearrangement inequalities

First, we denote the decreasing rearrangement of \( f \) on \( \mathbb{Z}^n \) by \( f^* \). We also denote \( f^{**}(n) := \frac{1}{n} \sum_{k=1}^{n} f^*(k) \).

Lemma 2.1. Let functions \( f, g, \) and \( K \) are defined on \( \mathbb{Z}^n \); then

\[ \sum_{k \in \mathbb{Z}} g(k)(K * f)(k) \leq 2 \sum_{r=1}^{\infty} rg^{**}(r)f^{**}(r)K^{**}(r). \] (2.1)

Proof. From \( f^{**}(n) = \sup_{|e|=n} \frac{1}{|e|} \sum_{s \in e} |f(s)| \) (see [BS, Ch. 2, §3]) and the Hardy-Littlewood inequality [BS, p.44], we write

\[ \sum_{k \in \mathbb{Z}} g(k)(K * f)(k) \leq \sum_{r=1}^{\infty} g^*(r)(K * f)^{**}(r) \]

\[ \leq \sum_{r=1}^{\infty} g^*(r) \sup_{|e|=r} \sum_{m \in \mathbb{Z}} |f(m)| \frac{1}{|e|} \sum_{s \in e} |K(s - m)| \]

\[ \leq \sum_{r=1}^{\infty} g^*(r) \sup_{|e|=r} \sum_{m=1}^{\infty} f^*(m) \left( \frac{1}{|e|} \sum_{s \in e} |K(s - \cdot)| \right)^{**} (m) \]

\[ \leq \sum_{r=1}^{\infty} g^*(r) \sup_{|e|=r} \sum_{m=1}^{\infty} f^*(m) \left( \sup_{|\omega|=m} \frac{1}{|e|} \frac{1}{|\omega|} \sum_{t \in \omega} \sum_{s \in e} |K(s - t)| \right) \]

\[ \leq \sum_{r=1}^{\infty} g^*(r) \sum_{m=1}^{\infty} f^*(m) \left( \sup_{|e|=r} \sup_{|\omega|=m} \frac{1}{|e|} \frac{1}{|\omega|} \sum_{t \in \omega} \sum_{s \in e} |K(s - t)| \right). \]
We consider
\[ \Phi(r, m) = \sup_{|e| = r, |\omega| = m} \sup_{e \subseteq \mathbb{Z}} \sup_{\omega \subseteq \mathbb{Z}} \sum_{t \in \omega} \sum_{s \in e} |K(s - t)|. \]

If \( r \leq m \), then
\[ \Phi(r, m) \leq \sup_{|e| = r, |\omega| = m} \sup_{e \subseteq \mathbb{Z}} \sup_{\omega \subseteq \mathbb{Z}} \sum_{s \in e} \sum_{t \in \omega} |K(s - t)| = K^{**}(m) \]
and if \( m \leq r \), then
\[ \Phi(r, m) \leq \sup_{|e| = r, |\omega| = m} \sup_{e \subseteq \mathbb{Z}} \sup_{\omega \subseteq \mathbb{Z}} \sum_{s \in e} \sum_{t \in \omega} |K(s - t)| = K^{**}(r). \]

Hence, we get
\[ \Phi(r, m) \leq K^{**}(\max\{r, m\}). \]

Therefore,
\[ \sum_{k \in \mathbb{Z}} g(k) (K * f)(k) \leq \sum_{r=1}^{\infty} g^*(r) \sum_{m=1}^{r} f^*(m) K^{**}(\max\{r, m\}) \]
\[ = \sum_{r=1}^{\infty} g^*(r) K^{**}(r) \sum_{m=1}^{r} f(m)^* \]
\[ + \sum_{r=1}^{\infty} g^*(r) \sum_{m=r+1}^{\infty} f^*(m) K^{**}(m) \]
\[ = \sum_{r=1}^{\infty} rg^*(r) K^{**}(r) f^**(r) + \sum_{m=1}^{\infty} f^*(m) K^{**}(m) \sum_{r=1}^{m} g^*(r) \]
\[ \leq 2 \sum_{r=1}^{\infty} rg^{**}(r) K^{**}(r) f^{**}(r). \]

The proof is complete. \( \Box \)

The continuous analogue of the previous lemma is the following result.

**Lemma 2.2.** Let \( f \) and \( g \) be measurable functions on \([0, d]\) and \( K \) be measurable on \([-d, d]\). Then

\[ (2.2) \quad \int_0^d g(y) \int_0^d f(x) K(y - x) \, dx \, dy \leq 2 \int_0^d tg^{**}(t) f^{**}(t) \left( \sup_{|e| = d} \frac{1}{|e|} \int_{|e| = 1} |K(x)| \, dx \right) dt. \]
Proof. Similarly to the proof of Lemma 2.1, we have

\[
\int_{0}^{d} g(y) (K * f)(y) dy \leq \int_{0}^{d} g^*(s) \int_{0}^{d} f^*(t) \sup_{e \in \mathcal{E}([0,d])} \frac{1}{|e|} \int_{\omega} |K(y - x)| dx dy
\]

\[
= \int_{0}^{d} g^*(s) \int_{0}^{d} f^*(t) \Phi(s, t) dt ds.
\]

Further, for \(s \leq t\), we get

\[
\Phi(s, t) \leq \sup_{e \in \mathcal{E}([0,d])} \frac{1}{|e|} \int_{e} |K(y)| dy.
\]

and for \(s \geq t\),

\[
\Phi(s, t) \leq \sup_{e \in \mathcal{E}([-d,d])} \frac{1}{|e|} \int_{e} |K(y)| dy.
\]

Finally, as in the proof of Lemma 2.1, we have

\[
\int_{0}^{d} g(y) (K * f)(y) dy \leq 2 \int_{0}^{d} t g^{**}(t) f^{**}(t) \sup_{e \in \mathcal{E}([-d,d])} \frac{1}{|e|} \int_{e} |K(x)| dx. \quad \square
\]

3. Proof of upper bound for \(\|A\|_{L_p \to L_q}\)

Let \(d > 0\), \(I = [0,d)\), and \(T_I = \{(md, (m+1)d)\}_{m \in \mathbb{Z}}\) be the corresponding partition of \(\mathbb{R}\). For a locally integrable function \(K(x)\) we put \(K(x) = K_1(x,d) + K_2(x,d)\), where

\[
K_1(x,d) = \begin{cases} 
K(x), & x \in (2md, (2m+1)d], \quad m \in \mathbb{Z} \\
0, & x \in ((2m-1)d, 2md], \quad m \in \mathbb{Z}
\end{cases}
\]

and

\[
K_2(x,d) = \begin{cases} 
0, & x \in (2md, (2m+1)d], \quad m \in \mathbb{Z} \\
K(x), & x \in ((2m-1)d, 2md], \quad m \in \mathbb{Z}
\end{cases}
\]

Then we write the convolution operator \(Af = f * K\) as \(A = A_1 + A_2\), where \(A_i f = f * K_i, \quad i = 1, 2\), we have

\[
\|A\|_{L_p \to L_q} \leq 2 \max_{i=1,2} \|A_i\|_{L_p \to L_q}. \tag{3.1}
\]
Let \( d > 0 \) for \( k \in \mathbb{Z} \) and \( x \in [0, d] \), we denote
\[
\tilde{f}(x, k) := f(x + kd),
\]
\[
\tilde{g}(x, k) := g(x + kd),
\]
\[
\tilde{K}_i(x, k) := K_i(x + kd).
\]

We are going to estimate the following quantity
\[
J_i := \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x) K_i(y - x) \, dx \, dy, \quad i = 1, 2.
\]

Let us write it as follows
\[
J_i = \sum_{k \in \mathbb{Z}} \int_{0}^{d} g(y + kd) \sum_{m \in \mathbb{Z}} \int_{0}^{d} f(x + md) K_i((y - x) + (k - m)d) \, dx \, dy.
\]

To estimate this functional, we first use Lemma 2.2:
\[
J_i \leq 2 \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \int_{0}^{d} \tilde{f}^{(**)}_1(t, m) \tilde{g}^{(**)}_1(t, k) \sup_{e \subset [-d, d]} \frac{1}{|e|} \int_{e} \tilde{K}_i(x, k - m) \, dx \, dt
\]
\[
= 2 \int_{0}^{d} t \left( \sum_{k \in \mathbb{Z}} \tilde{g}^{(**)}_1(t, k) \sum_{m \in \mathbb{Z}} \tilde{f}^{(**)}_1(t, m) \sup_{e \subset [-d, d]} \frac{1}{|e|} \int_{e} \tilde{K}_i(x, k - m) \, dx \right) \, dt,
\]
where
\[
\tilde{f}^{(**)}_1(t, m) = \frac{1}{t} \int_{0}^{t} \tilde{f}^{**}(t, m) \, dt, \quad m \in \mathbb{Z},
\]
\[
\tilde{g}^{(**)}_1(t, k) = \frac{1}{t} \int_{0}^{t} \tilde{g}^{**}(t, k) \, dt, \quad k \in \mathbb{Z},
\]
and \( \tilde{f}^{**}(t, m), \tilde{g}^{**}(t, k) \) are decreasing rearrangements of \( \tilde{f}(x, m), \tilde{g}(x, k) \) with respect to \( x \) and with fixed \( m \) and \( k \), correspondingly.

Applying now Lemma 2.1, we get
\[
J_i \leq 4 \int_{0}^{d} t \sum_{s=1}^{\infty} s \tilde{f}^{**}(t, s) \tilde{g}^{**}(t, s) \left( \sup_{|\omega| = s} \frac{1}{|\omega|} \sum_{e \in [-d, d]} \sup_{|e| = t} \frac{1}{|e|} \int_{e} \tilde{K}_i(x, m) \, dx \right) \, dt
\]
\[
\equiv 4 \int_{0}^{d} t \sum_{s=1}^{+\infty} s \tilde{f}^{**}(t, s) \tilde{g}^{**}(t, s) F_d(t, s; K_i) \, dt,
\]
where

\[ \tilde{f}^{**}(t, s) = \frac{1}{s} \sum_{l=1}^{s} \left( \tilde{f}^{(**)}(t, \cdot) \right)_{l}^{*2}, \]

\[ \tilde{g}^{**}(t, s) = \frac{1}{s} \sum_{l=1}^{s} \left( \tilde{g}^{(**)}(t, \cdot) \right)_{l}^{*2}. \]

Then writing

\[ (ts)\tilde{g}^{**}(t, s)\tilde{f}^{**}(t, s)F_{d}(t, s; K_{i}) \leq \left( (ts)\frac{1}{p} - \frac{1}{q} \right)\tilde{f}^{**}(t, s) \left( \sup_{s \in \mathbb{N}} (ts)^{\frac{1}{p}} F_{d}(t, s; K_{i}) \right) \]

and using Hölder’s inequality with parameters \( q \) and \( q' \) and the fact that \( L^{pq} \hookrightarrow L_{pq} \) for \( q \leq q_1 \), we get

\[
\int_{0}^{d} \sum_{s=1}^{\infty} (ts)^{1 - \left( \frac{1}{p} - \frac{1}{q} \right)} F_{d}(t, s; K_{i}) \left( \sum_{s \in \mathbb{N}} \int_{0}^{d} (\tilde{g}^{**}(t, s))^{q'} dt \right)^{1/q'} \left( \sum_{s \in \mathbb{N}} \int_{0}^{d} (\tilde{f}^{**}(t, s))^{p} dt \right)^{1/p}.
\]

Then by Hardy’s inequality,

\[
\left| \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x)K_{i}(y - x)\,dx\,dy \right| \leq C \sup_{s \in \mathbb{N}} (ts)^{1 - \left( \frac{1}{p} - \frac{1}{q} \right)} F_{d}(t, s; K_{i}) \| f \|_{L_{p}} \| g \|_{L_{q}'}.
\]

Thus,

\[
(3.3) \quad \| A_{i} \|_{L_{p} \rightarrow L_{q}} \leq C \sup_{s \in \mathbb{N}} (ts)^{1 - \left( \frac{1}{p} - \frac{1}{q} \right)} F_{d}(t, s; K_{i}), \quad i = 1, 2.
\]

Note that by definition, \( K_{1} \) and \( K_{2} \) satisfy

\[ \text{supp } \tilde{K}_{1}(x, s) \subset [-d, 0] \times \mathbb{Z}, \quad \text{supp } \tilde{K}_{2}(x, s) \subset [0, d] \times \mathbb{Z}. \]

Therefore,

\[
\sup_{e \subset [-d, d], \|e\|=t} \frac{1}{|e|} \int_{e} |K_{1}(x, k)| \,dx = \sup_{e \subset [-d, 0], \|e\|=t} \frac{1}{|e|} \int_{e} |\tilde{K}_{1}(x, k)| \,dx, \quad k \in \mathbb{Z}
\]

and

\[
\sup_{e \subset [-d, d], \|e\|=t} \frac{1}{|e|} \int_{e} |K_{2}(x, k)| \,dx = \sup_{e \subset [0, d], \|e\|=t} \frac{1}{|e|} \int_{e} |\tilde{K}_{2}(x, k)| \,dx, \quad k \in \mathbb{Z}.
\]
Then

\[
\sup_{0 < t \leq d, s \in \mathbb{N}} (ts)^{1-(\frac{1}{p}-\frac{1}{q})} F_d(t, s; K_1) = \sup_{0 < t \leq d, |\omega| = s} \sup_{s \in \mathbb{N}} \frac{1}{(ts)^{\frac{1}{p}-\frac{1}{q}}} \sum_{m \in \omega} \sup_{|\epsilon| = 1} \int \epsilon |\tilde{K}_1(x, m)| \, dx.
\]

For any \(m \in \mathbb{Z}\) and \(t \in (0, d]\) we find \(e_{m,t} \subset [-d, 0]\) such that \(|e_{m,t}| = t\) and

\[
\sup_{|\epsilon| = 1} \int \epsilon |\tilde{K}_1(x, m)| \, dx \leq 2 \int_{e_{m,t}} \epsilon |\tilde{K}_1(x, m)| \, dx
\]

\[
= 2 \int_{e_{m,t} + md} |K_1(x + md)| \, dx = 2 \int_{e_{m,t} + md} |K(x)| \, dx.
\]

The set \(\eta_m = e_{m,t} + md\) of measure \(t\) for different \(m\) belongs to different elements of \(T_d = \{nd, (n+1)d\}_{n \in \mathbb{Z}}\). So, for \(0 < t \leq d\) and \(r \in \mathbb{N}\) we have

\[
\sup_{|\omega| = s} \sup_{s \in \mathbb{N}} \int \epsilon |\tilde{K}_1(x, m)| \, dx \leq 2 \sup_{e \in \mathbb{U}(0,d]} \frac{1}{|\epsilon|^{\frac{1}{p}-\frac{1}{q}}} \int_{e} |K(x)| \, dx.
\]

Therefore, we obtain

\[
\sup_{0 < t \leq d, s \in \mathbb{N}} (ts)^{1-(\frac{1}{p}-\frac{1}{q})} F_d(t, s; K_1) \leq 2 \sup_{e \in \mathbb{U}(0,d]} \frac{1}{|\epsilon|^{\frac{1}{p}-\frac{1}{q}}} \int_{e} |K(x)| \, dx
\]

and, similarly,

\[
\sup_{0 < t \leq d, s \in \mathbb{N}} (ts)^{1-(\frac{1}{p}-\frac{1}{q})} F_d(t, s; K_2) \leq 2 \sup_{e \in \mathbb{U}(0,d]} \frac{1}{|\epsilon|^{\frac{1}{p}-\frac{1}{q}}} \int_{e} |K(x)| \, dx.
\]

Combining this with (3.1) and (3.3), we get

\[
\|A\|_{L_p \to L_q} \leq C \sup_{e \in \mathbb{U}(0,d]} \frac{1}{|\epsilon|^{\frac{1}{p}-\frac{1}{q}}} \int_{e} |K(x)| \, dx
\]

and using an arbitrary choice of \(d > 0\),

\[
\|A\|_{L_p \to L_q} \leq C \sup_{d > 0} \sup_{e \in \mathbb{U}(0,d]} \frac{1}{|\epsilon|^{\frac{1}{p}-\frac{1}{q}}} \int_{e} |K(x)| \, dx
\]

with a constant \(C\) depending on \(p\) and \(q\). Since the norms of operators \(A f = K * f\) and \(A_t f = K_t * f\), where \(K_t(x) = K(x + t), \ t \geq 0\) coincide, the last estimate implies

\[
\|A\|_{L_p \to L_q} \leq C \sup_{I} \sup_{e \in \mathbb{U}(I)} \frac{1}{|\epsilon|^{\frac{1}{p}-\frac{1}{q}}} \int_{e} |K(x)| \, dx.
\]

To finish this proof, it is sufficient to show the following

**Lemma 3.1.** Let \(0 < \gamma \leq 1\) and \(K\) be locally integrable. Then for any \(e \in \mathbb{U}(I)\) there exists \(e' \in \mathbb{U}(I)\) such that

\[
\frac{1}{|\epsilon|^{\gamma}} \int_{e} |K(x)| \, dx \leq 2^{2-\gamma} \frac{1}{|\epsilon'|^{\gamma}} \int_{e'} |K(x)| \, dx.
\]
\textbf{Proof.} Since \( e \in \mathcal{F}(I) \) we have \( e = \bigcup_{k=1}^{m} \omega_k \), where \( |\omega_k| = \omega < d, k = 1, m \) and \( \omega_k \) belong to different elements of \( T_I = \{ I + kd \}_{r \in \mathbb{Z}} \).

For any \( \omega_k \) let us define
\[
\omega_k^1 := \left\{ x \in \omega_k : K(x) \geq 0 \right\} \quad \text{and} \quad \omega_k^2 := \left\{ x \in \omega_k : K(x) < 0 \right\}.
\]

Then
\[
\int_{\omega_k} |K(x)| dx = \int_{\omega_k^1} K(x) dx - \int_{\omega_k^2} K(x) dx \leq 2 \max \left\{ \left| \int_{\omega_k^1} K(x) dx \right|, \left| \int_{\omega_k^2} K(x) dx \right| \right\}.
\]

We can assume that
\[
\left| \int_{\omega_k^1} K(x) dx \right| \geq \left| \int_{\omega_k^2} K(x) dx \right|.
\]

Let us consider two cases: \( |\omega_k^1| \geq \frac{\omega}{2} \) and \( |\omega_k^2| < \frac{\omega}{2} \). In the first case, there exists \( \tilde{\omega}_k \subset \omega_k \) such that \( |\tilde{\omega}_k| = \frac{\omega}{2} \) and
\[
2 \left| \int_{\tilde{\omega}_k} K(x) dx \right| \geq \left| \int_{\omega_k} K(x) dx \right|.
\]

In the second case, \( |\omega_k^1| < \frac{\omega}{2} \) and there exist \( \eta_k^1 \) and \( \eta_k^2 \) such that \( |\eta_k^1 \cap \eta_k^2| = 0 \), \( \eta_k^1 \cup \eta_k^2 = \omega_k^2 \), and \( |\eta_k^1| = \frac{|\omega_k^1|}{2} \). Since \( K(x) \) keeps its sign on \( \omega_k^2 \), we have
\[
\left| \int_{\omega_k^1} K(x) dx \right| \geq \left| \int_{\eta_k^1} K(x) dx \right| + \left| \int_{\eta_k^2} K(x) dx \right| \geq 2 \min \left( \left| \int_{\eta_k^1} K(x) dx \right|, \left| \int_{\eta_k^2} K(x) dx \right| \right) = 2 \left| \int_{\omega_k^1} \eta_k^{i_0} K(x) dx \right|.
\]

Here \( \eta_k^{i_0} \) are sets where the infimum is attained. Then we consider \( \eta_k \subset \eta_k^{i_0} \) such that \( |\eta_k| = \frac{\omega}{2} - |\omega_k^1| \).

Let now \( \tilde{\omega}_k = \eta_k \cup \omega_k^1 \), then \( |\tilde{\omega}_k| = \frac{\omega}{2} \) and
\[
\left| \int_{\tilde{\omega}_k} K(x) dx \right| = \left| \int_{\omega_k^1} K(x) dx \right| + \left| \int_{\eta_k} K(x) dx \right| \geq \left| \int_{\omega_k^1} K(x) dx \right| - \left| \int_{\eta_k} K(x) dx \right| \geq \frac{1}{2} \left| \int_{\omega_k^1} K(x) dx \right|.
\]
Therefore, in both cases, we have
\[ \int_{\omega_k} |K(x)| \, dx \leq 2 \left| \int_{\omega_1} K(x) \, dx \right| \leq 4 \left| \int_{\omega_k} K(x) \, dx \right|. \]

Suppose
\[ J_+ = \{ k : \int_{\omega_k} K(x) \, dx \geq 0 \}, \quad J_- = \{ k : \int_{\omega_k} K(x) \, dx \leq 0 \}; \]
then
\[ 2 \max \left\{ \left| \int_{\bigcup_{k \in J_+} \omega_k} K(x) \, dx \right|, \left| \int_{\bigcup_{k \in J_-} \omega_k} K(x) \, dx \right| \right\} \geq \sum_{k=1}^{m} \left| \int_{\omega_k} K(x) \, dx \right| \geq \frac{1}{4} \int |K(x)| \, dx. \]

Taking as \( e' \) the set \( \bigcup_{k \in J_+} \tilde{\omega}_k \) or \( \bigcup_{k \in J_-} \tilde{\omega}_k \), where the maximum is attained, we get
\[ \frac{1}{|\epsilon'|^{\gamma}} \int_{e'} |K(x)| \, dx \leq 2^3 \frac{1}{|\epsilon'|^{\gamma}} \left| \int_{e'} K(x) \, dx \right| \leq 2^{3-\gamma} \frac{1}{|\epsilon'|^{\gamma}} \left| \int_{e'} K(x) \, dx \right|. \]

By construction, \( \tilde{\omega}_k \subset \omega_k \) and \( |\tilde{\omega}_k| = \frac{d}{2} \), and therefore \( e' \in \mathcal{U}(I) \). \[ \square \]

4. **Proof of lower bound for \( \|A\|_{L_p \rightarrow L_q} \)**

Let \( 1 < p < q < \infty \), \( \frac{1}{r} = 1 - \left( \frac{1}{p} - \frac{1}{q} \right) \), and \( Af = K * f \) is bounded from \( L_p(\mathbb{R}) \) in \( L_q(\mathbb{R}) \). We are going to prove that for any number \( d \) and an interval \( I, |I| = d \), there holds
\[ \sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| \leq c(p, q) \|A\|_{L_p \rightarrow L_q}, \tag{4.1} \]
where the collection \( \mathcal{L}(I) \) is given by (1.4). We define \( \mathcal{L}'(I) \subset \mathcal{L}(I) \) as follows
\[ \mathcal{L}'(I) = \left\{ e = \bigcup_{k=0}^{m} [(a, b] + kd) : m \in \mathbb{N}, [a, b] \subset I, b - a \leq d/2 \right\}. \]

Note that for any locally summable function \( K(x) \) we have
\[ \sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| \leq \sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| \]
\[ \leq 2^{1/r} \sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right|. \]
Indeed, the left-hand side inequality is clear since $\mathcal{L}'(I) \subset \mathcal{L}(I)$. To prove the right-hand side, we consider $e \in \mathcal{L}(I)$, that is, $e = \bigcup_{i=0}^m ([a, b] + id) = \bigcup_{i=0}^m ([a, \frac{a+b}{2}] + id) \cup \bigcup_{i=0}^m ([\frac{a+b}{2}, b] + id) = e_1 \cup e_2$. Clearly, $|e_i| = |e|/2$, $e_i \in \mathcal{L}'(I)$, $i = 1, 2$. Therefore,

$$\frac{1}{|e|^{1/r'}} \left| \int_e K(x)dx \right| \leq \frac{2}{|e|^{1/r'}} \max_{i=1,2} \left| \int_{e_i} K(x)dx \right|$$

$$= 2^{1-1/r'} \max_{i=1,2} \frac{1}{|e_i|^{1/r'}} \left| \int_{e_i} K(x)dx \right|$$

$$\leq 2^{1/r'} \sup_{e \in \mathcal{L}'(I)} \frac{1}{|e|^{1/r'}} \left| \int_e K(x)dx \right|.$$ 

Hence, it is sufficient to verify

$$\sup_{e \in \mathcal{L}'(I)} \frac{1}{|e|^{1/r'}} \left| \int_e K(x)dx \right| \leq c \|A\|_{L_p \rightarrow L_q}.$$ 

Let us first assume that $K$ is bounded, that is, $|K(x)| \leq D$, $x \in \mathbb{R}$. For $s > 0$ we define

$$\alpha_s = \sup_{e \in \mathcal{L}'(I)} \frac{1}{|e|^{1/r'}} \left| \int_e K(x)dx \right|.$$ 

This is well-defined since for any $e \in \mathcal{L}'(I)$ and $|e| \leq s$ we get

$$\frac{1}{|e|^{1/r'}} \left| \int_e K(x)dx \right| \leq D \frac{1}{|e|^{1-1/r'}} \leq D s^{1-1/r'}.$$ 

Then we consider $e_0 \in \mathcal{L}'(I)$, $|e_0| \leq s$ such that

$$\frac{1}{|e_0|^{1/r'}} \left| \int_{e_0} K(x)dx \right| \geq \frac{\alpha_s}{2}.$$ 

Since the convolution is translation invariant, then we assume that $e_0$ is of form

$$e_0 = \bigcup_{i=0}^m ([0, b] + id),$$

where $b \leq d/2$, $m \in \mathbb{N} \cup \{0\}$.

Let us take $0 < \delta < \frac{1}{2}$ to be specified later. We define the following sets $e_{1+\delta}$ and $e_\delta$:

$$e_{1+\delta} = \bigcup_{i=0}^{(1+\delta)m} ([0, (1+\delta)b] + id),$$

$$e_\delta = \bigcup_{i=0}^{[\delta m]} ([0, \delta b] + id).$$
Since \( e_0 \in \mathcal{L}(I) \), we have \( e_{1+\delta} \in \mathcal{L}(I) \). Then taking \( f_0 = \chi_{e_{1+\delta}} \), boundedness of the operator \( A \) implies

\[
\|K \ast f_0\|_{L^q} \leq \|A\|_{L^p \rightarrow L^q} \|f_0\|_{L^p}
\]

(4.2) 

\[
= \|A\|_{L^p \rightarrow L^q} |e_{1+\delta}|^{1/p} \leq 2 \|A\|_{L^p \rightarrow L^q} (1 + \delta)^{2/p} |e_0|^{1/p}.
\]

On the other hand,

\[
\|K \ast f_0\|_{L^q} = \left( \int_{-\infty}^{\infty} \left| \int_{e_{1+\delta}}^{d} K(x-y)dx \right|^q dy \right)^{1/q}
\]

\[
= \left( \sum_{j \in \mathbb{Z}} \int_{0}^{d} \left| \sum_{i=0}^{(1+\delta)m} \int_{0}^{(1+\delta)b} K((i-j)d + (x-y))dx \right|^q dy \right)^{1/q}
\]

\[
\geq \left( \sum_{j=0}^{[\delta b]} \left| \sum_{i=0}^{(1+\delta)b} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right|^q dy \right)^{1/q}
\]

where \( e_i \in \mathcal{L}(I) \) such that \( |e_i| \leq 2\delta |e_0|, i = 1, 2, 3, 4. \)

We put \( \delta = \left(2(16^r')\right)^{-1} < \frac{1}{2} \). Then \( |e_i| < |e_0| \leq s \) and

\[
\frac{1}{|e_0|^{1/r'}} \left| \int_{e_0} K(x)dx \right| \geq \frac{\alpha s}{2} \geq \frac{1}{2 |e_i|^{1/r'}} \int_{[e_i]} K(x)dx
\]
and therefore  
\[ \left| \int_{e_i} K(x) \, dx \right| \leq \frac{2 \left| e_i \right|^{1/r'}}{\left| e_0 \right|^{1/r'}} \left| \int_{e_0} K(x) \, dx \right|. \]

Taking into account \( |e_i| \leq 2\delta |e_0| \), we get  
\[ \| K * f_0 \|_{L_q} \geq \left( \int_{e_0} \left| \int_{e_0} K(x) \, dx \right| \left( 1 - 2 \sum_{i=1}^{4} \left( \frac{|e_i|}{|e_0|} \right)^{1/r'} \right) \right)^{1/q} \]

\[ \geq \left| e_\delta \right|^{1/q} \left| \int_{e_0} K(x) \, dx \right| \left( 1 - 8(2\delta)^{1/r'} \right) \]

\[ \geq \frac{1}{2} \delta^{2/q} |e_0|^{1/q} \left| \int_{e_0} K(x) \, dx \right|. \]

Using (4.2), we have  
\[ \| A \|_{L_p \rightarrow L_q} \geq C_{p,q} \frac{1}{|e_0|^{1/r'}} \left| \int_{e_0} K(x) \, dx \right| \geq \frac{C_{p,q}}{2} \sup_{e \in \mathbb{S}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| \]

Thus, for the bounded \( K \) and for any \( s > 0 \) we obtain  
\[ (4.3) \sup_{e \in \mathbb{S}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| \leq C \| A \|_{L_p \rightarrow L_q}, \]

where \( C \) depends on \( p \) and \( q \).

To prove this in a general case of locally integrable \( K \) not necessary bounded, we consider  
\[ K_N(x) = \begin{cases} N, & K(x) > N, \\ K(x), & K(x) \leq N \end{cases}, \quad N \in \mathbb{N} \]

and  
\[ K_{N,M}(x) = \begin{cases} K_N(x), & K(x) \geq -M, \\ M, & K(x) < -M \end{cases}, \quad N, M \in \mathbb{N}. \]

As we have proved before,  
\[ \sup_{e \in \mathbb{S}(I)} \frac{1}{|e|^{1/r'}} \left| \int_{e} K_{N,M}(x) \, dx \right| \leq C \| A_{N,M} \|_{L_p \rightarrow L_q}, \quad A_{N,M} f = K_{N,M} * f, \]

where a constant \( C \) does not depend on \( N \) and \( M \).

Noting that Banach-Steinhaus’ theorem implies \( \| A_{N,M} \|_{L_p \rightarrow L_q} \leq D (D > 0) \) for some \( D > 0 \) and using the monotonicity properties of \( K_{N,M} \), namely,  
\[ K_{N,1}(x) \geq K_{N,2} \geq \ldots \geq K_{N,M} \geq \ldots \]

and  
\[ K_1(x) \leq K_2(x) \leq \ldots K_N \leq \ldots, \]
we apply Levi’s theorem:
\[
\sup_{e \in \mathcal{L}(I)} \frac{1}{|e|^{1/r'}} \left| \int e K(x) dx \right| \leq cD < \infty.
\]
Finally, repeating the proof of (4.3), we arrive at required inequality (4.1).

We would like to mention that attempts have already been made at proving
the lower estimate for the convolution operator in [NS], although they require
stronger hypotheses than those used here.

5. Comparison with O’Neil and Stepanov’s inequalities

Let us first show that the right-hand side estimate in (1.5) implies both (1.2)
and (1.3). Indeed, it is known that ([BS, Ch. 2, §3])
\[
(5.1) \quad \sup_{t > 0} t^{1/r} K^*(t) \approx \sup_{0 < |e| < \infty} \frac{1}{|e|^{1/r'}} \int |K(x)| dx,
\]
and therefore
\[
\sup_{e \in \mathcal{U}(I)} \frac{1}{|e|^{1/r'}} \left| \int e K(x) dx \right| \leq C \sup_{t > 0} t^{1/r} K^*(t).
\]
Let \(1/r = 1 - (1/p - 1/q) < 1\), \(r' = r/(r-1)\), and let \(I\) be an interval with
\(|I| = 1\). Assume that \(e \in \mathcal{U}(I)\), that is, \(e = \bigcup_{s=1}^{m} w_s\) such that \(w_s \in I + i_s, |w_s| = |w_j| = g \leq 1, i_s \neq i_j\). Then
\[
\frac{1}{|e|^{1/r'}} \left| \int e K(x) dx \right| = \left| \frac{1}{(mg)^{1/r'}} \sum_{s=1}^{m} \int_{w_s} K(x) dx \right|
\]
\[
= \frac{1}{(mg)^{1/r'}} \left| \sum_{s=1}^{m} \int_{0}^{1} K(x + i_s) \chi_{w_s}(x + i_s) dx \right|
\]
\[
\leq \frac{1}{(mg)^{1/r'}} \sum_{s=1}^{m} \int_{0}^{1} |K(x + i_s) \chi_{w_s}(x + i_s)| dx
\]
\[
= \frac{1}{(mg)^{1/r'}} \sum_{s=1}^{m} \int_{0}^{g} \tilde{K}^*(t, i_s) dt
\]
\[
\leq \frac{1}{(mg)^{1/r'}} \sum_{s=1}^{m} \sup_{0 < t \leq 1} t^{1/r} \tilde{K}^*(t, i_s) \left( \int_{0}^{g} t^{-1/r} dt \right)\]
\[
\leq (r')^2 \sup_{n \in \mathbb{N}} \left( \sup_{0 < t \leq 1} t^{1/r} \tilde{K}^*(t, \cdot) \right)_{s}^*.
\]
Thus, (1.5) is stronger than either (1.2) or (1.3). We now give an example
capturing the difference between these estimates.
Example. Let \( 1 < p < q < \infty \) and \( 0 < 1/r = 1 - (1/p - 1/q) \). Define the function \( K(x) \) on \( \mathbb{R} \) as follows:

\[
K(x) = \begin{cases} 
2^{k/r}, & \text{for } x \in [-k, -k + 2^{-k}], \ k \in \mathbb{N}; \\
1, & \text{for } x \in [k, k + 1/k), \ k \in \mathbb{N}; \\
0, & \text{otherwise.}
\end{cases}
\]

This function satisfies

\[
\sup_{e \in \mathcal{U}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| < \infty,
\]

but

\[
\|K\|_{L^r,\infty} = \infty
\]

and

\[
\|K\|_{W(L^r,\infty([0,1]),L^\infty(\mathbb{Z}))} = \sup_{n \in \mathbb{N}} n^{1/r} \left( \sup_{0 \leq t \leq 1} t^{1/r} K^*(t, \cdot) \right)_n = \infty.
\]

Indeed, let us show (5.2). Let \( K_+(x) = K(x) \chi_{[0,\infty)}(x) \), \( K_-(x) = K(x) \chi_{(-\infty,0)}(x) \), then \( K_+(x) + K_-(x) = K(x) \) and therefore,

\[
\sup_{e \in \mathcal{M}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_{e} K(x) \, dx \right| 
\leq \sup_{e \in \mathcal{M}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_{e} K_+(x) \, dx \right| + \sup_{e \in \mathcal{M}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_{e} K_-(x) \, dx \right|.
\]

Let \( e \in \mathcal{U}([0,1]) \). Then \( e = \cup_{k \in u} w_k \), where \( |w_k| = w < 1 \) and \( u \subseteq \mathbb{Z} \), \( |u| = m \).

We have

\[
\frac{1}{|e|^{1/r'}} \int_{e} K_+(x) \, dx = \frac{1}{|m|^{1/r'}} \frac{1}{w^{1/r'}} \sum_{k \in u} \left| \int_{w_k} K_+(x) \, dx \right|
= \frac{1}{m^{1/r'}} \frac{1}{w^{1/r'}} \sum_{n=1}^{m} \int_{0}^{w} K_+(x + n) \, dx
= \frac{1}{(wm)^{1/r'}} \left( \sum_{n=1}^{1/w} \int_{0}^{w} K_+(x + n) \, dx + \sum_{n=1/w}^{m} \int_{0}^{1/n} K_+(x + n) \, dx \right)
= \frac{1}{(wm)^{1/r'}} \left( \sum_{n=1}^{1/w} n + \sum_{n=1/w}^{m} \frac{1}{n} \right)
\leq \frac{2}{(wm)^{1/r'}} \left( 1 + \ln(mw) \right) \leq 2r'.
\]
Further,
\[
\frac{1}{|e|^{1/r'}} \int_e K_-(x) \, dx = \frac{1}{|m|^{1/r'}} \frac{1}{w^{1/r'}} \sum_{k \in u} \left| \int_{w_k} K_-(x) \, dx \right|
\]
\[
= \frac{1}{(mw)^{1/r'}} \frac{1}{w^{1/r'}} \sum_{n \in u} \int_0^w K_-(x + n) \, dx
\]
\[
\leq \frac{1}{(wm)^{1/r'}} \left( \sum_{n \in u, |n| < \log_2 \frac{1}{w}} \int_0^w K_-(x - |n|) \, dx + \sum_{n \in u, |n| \geq \log_2 \frac{1}{w}} \int_0^{2^{-|n|}} K_-(x - |n|) \, dx \right)
\]
\[
= \frac{1}{(wm)^{1/r'}} \left( \sum_{n \in u, |n| < \log_2 \frac{1}{w}} 2^{|n|/r'} w + \sum_{n \in u, |n| \geq \log_2 \frac{1}{w}} 2^{-|n|/r'} \right)
\]
\[
\leq 2 \frac{1}{(wm)^{1/r'}} \left( w^{1/r'} + w^{1/r} \right) \leq 4.
\]
Combining these estimates, we get
\[
\sup_{e \in \mathcal{H}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K(x) \, dx \right| \leq 4 + 2r'.
\]
To show (5.3), we note that \( K^*_+(t) \equiv 1 \). Hence,
\[
\sup_{t>0} t^{1/r} K^*(t) \geq \sup_{t>0} t^{1/r} K^*_+(t) = \infty.
\]
To show (5.4), we note
\[
\|K\|_{W(L_{r,\infty}, [0,1], L_{r,\infty}(\mathbb{Z}))} \geq \|K-\|_{W(L_{r,\infty}, [0,1], L_{r,\infty}(\mathbb{Z}))}
\]
\[
= \sup_{n \in \mathbb{N}} n^{1/r} \left( \sup_{0 < t \leq 1} t^{1/r} (\tilde{K}_N^-) \cdot(t, n) \right)_n
\]
\[
= \sup_{n \in \mathbb{N}} n^{1/r} \left( \sup_{0 < t \leq 2^{-n}} t^{1/r} 2^n \right)
\]
\[
= \sup_{n \in \mathbb{N}} n^{1/r} = \infty.
\]
6. Acknowledgements

This research was supported by the RFFI 09-01-00175, NSH-2787.2008.1, and MTM 2008-05561-C02-02. The paper was started when the first and the second authors were staying at the Centre de Recerca Matemática (Barcelona).

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