Nonuniform Time-step Runge-Kutta Discontinuous Galerkin Method for Computational Aeroacoustics

Li Liu* and Xiaodong Li†
Beihang University, Beijing, 100191, People Republic of China

Fang Q. Hu‡
Old Dominion University, Norfolk, Virginia 23529, USA

In computational aeroacoustics (CAA) simulations, discontinuous Galerkin space discretization (DG) in conjunction with Runge-Kutta time integration (RK), which is so called Runge-Kutta discontinuous Galerkin method (RKDG), has been an attractive alternative to the finite difference based high-order numerical approaches. However, when it comes to complex physical problems, especially the ones involving irregular geometries, an expensive computational cost are usually required for time-accurate solution, because the time step size of an explicit RK scheme is limited by the smallest grid size in the computational domain. For computational efficiency, high-order RK method with nonuniform time step sizes on nonuniform meshes is developed in this paper. On the elements neighboring the interfaces of grids with different time step sizes, the values at intermediate stages of RK time integration are coupled suitably to realize the stable communication of solutions at those interfaces with minimal dissipation and dispersion errors. A linear coupling procedure is described based upon the general form of a p-stage RK scheme, and also extended to the high-order RK schemes frequently used in simulation of fluid flow and acoustics, including the third order TVD scheme, and low-storage low dissipation and low dispersion schemes. In addition, an eigenvalue analysis of nonuniform time-step RK integration combined with DG method on a nonuniform grid is carried out for the stability property. For verification, numerical experiments on one-dimensional and two-dimensional linear problems are conducted to illustrate the stability and accuracy of proposed nonuniform time-step RKDG scheme. Application to a one-dimensional nonlinear problem is also investigated.

I. Introduction

With many superior features, Runge-Kutta Discontinuous Galerkin (RKDG) method, which adopts discontinuous Galerkin approximation in space and explicit Runge-Kutta integration in time, has been widely applied in modeling advection-dominant physical phenomena.\textsuperscript{1,6,7,13,15,21} For aeroacoustic calculations, in order to simulate the acoustic waves propagating from near field to far field, a long time integration with low dissipation and low dispersion errors is usually required in a large space domain. In this regard, a variety of high-order Runge-Kutta methods have been applied, for instance, the classical third and fourth order methods,\textsuperscript{4,16} the third-order TVD methods,\textsuperscript{11} and the optimized schemes with minimized dispersion and dissipation errors.\textsuperscript{2,5,14} Typically they are designed for a uniform time step size in the entire computational domain. However, when it comes to realistic complex problems, especially the ones involving irregular geometries and the ones involving a large disparity of physical scales, nonuniform grids are usually adopted for modeling resolution and simulation effectiveness. For such cases, the time step size would be limited by the smallest grid size due to stability requirements. Consequently, with a uniform time step for the entire
computational domain, excessive computational time would be required for obtaining time-dependant numerical solutions, which results in a barrier for many practical applications. Clearly, due to the popularity of Runge-Kutta schemes, there is a need for the extension of Runge-Kutta methods to nonuniform time steps.

In recent literatures, efforts have been made to achieve time marching with variable time step sizes in nonuniform grids. In Ref. 8 and Ref. 22, Adam-Bashforth type time integration schemes with multiple time step sizes are developed in Cartesian and curvilinear grid in conjunction with the dispersion relation preserving (DRP) finite difference schemes. Recently, the Cauchy-Kovalevskaja (CK) procedures, which are sometimes called the Lax-Wendroff procedures, have been developed with arbitrary ratio of time-step sizes in the framework of the ADER schemes (Arbitrary high order schemes using derivatives), such as the ADER finite volume (ADER-FV), and the ADER discontinuous Galerkin (ADER-DG) and the ADER finite difference (ADER-FD) schemes, and the space-time expansion DG scheme (STE-DG). In the present work, Runge-Kutta method with nonuniform time step on nonuniform meshes is considered, where the spatial discretization is carried out by the Discontinuous Galerkin (DG) method. For correct communication of solutions between meshes with different time step sizes, the coupling of values at intermediate stages of Runge-Kutta time integration should be implemented synchronously for the elements neighboring the interfaces with low dissipation and low dispersion errors. Formulation of the coupling procedures for multi-stage Runge-Kutta integration will be presented based on the linearity of the equation. It makes the time marching with local step size available for the Runge-Kutta family of time integration schemes, and is valid for an arbitrary time step ratio. An approach for optimization of the coupling coefficients with minimal dissipation and dispersion errors is provided. In addition, the stability of the scheme developed is investigated. Numerical validations in one- and two-dimensional cases are performed to demonstrate the stability and accuracy of the scheme for both linear and nonlinear cases.

The remainder of the paper is organized as follows. Discontinuous Galerkin spatial discretization is reviewed in Section 2. In Section 3, the linear coupling procedure that is capable of handling nonuniform time-step integration on a nonuniform grid is formulated based upon the general form of a p-stage Runge-Kutta scheme, and an optimization of coupling coefficients with low dissipation and low dispersion errors will be discussed. A stability analysis for the nonuniform time-step RKDG scheme is given in Section 4. Finally, the numerical experiments are conducted to validate the method on one- and two-dimensional problems in Section 5. Section 6 has the conclusions.

II. The discontinuous Galerkin spatial discretization

For convenience of discussion, we first briefly review a Discontinuous Galerkin (DG) semi-discretization for partial differential equations (PDE). Consider a system of hyperbolic PDE’s of the form given below in a domain $\Omega \subseteq \mathbb{R}^d$:

$$\frac{\partial U}{\partial t} + \nabla \cdot F = S$$

where the solution $U(x,t)$ is time dependant, $S$ is the source term. The flux $F = \{f_i\}_{i=1,\ldots,d}$, and $d$ is the number of space dimensions. For linear problems, $f_i = A_i U$, and $A_i$ are Jacobian matrices.

In the DG method, the computational domain of interest $\Omega$ is divided into non-overlapping elements, $\Omega_k, k = 1, 2, \ldots, N_e$. Different from the traditional Galerkin method, the approximation space $V_h$ here could be discontinuous across element interfaces. The space $V_h$ is obtained by defining on each element $\Omega_k$ the local polynomial space $P(\Omega_k)$ which contains only spatial functions in the semi-discrete formulation

$$V_h = \{v \in L^1(\Omega) : v|_{\Omega_k} \in P(\Omega_k)\}$$

Assume the local basis set in $\Omega_k$ be $B_k = \{v_k^l\}_{l=1,\ldots,N}$, which is usually chosen to be the same for all elements; and $N$ is the degree of freedom in $\Omega_k$ that is dependent on the number of space dimensions $d$ and the order of scheme $P$. Then the local approximate solution $U_h$ in $\Omega_k$ can be expressed in an expansion of the local basis set as

$$U_h|_{\Omega_k}(x,t) = \sum_{l=0}^{N-1} C_k^l(t) v_k^l(x)$$

where $C_k^l$ are the expansion coefficients of the solution in $B_k$. It has been shown that the accuracy of the method can reach $P + 1/2$ on the general triangular grid, and even $P + 1$ on some Cartesian grid and some semi-uniform triangular grid when the order of polynomial space is $P$.17-20
Via an integration by parts, the weak formulation of DG spatial discretization in $\Omega_k$ can be obtained as

$$
\int_{\Omega_k} v^v\partial U^k \frac{\partial}{\partial t} \, dx + \int_{\partial\Omega_k} v^v F^R \cdot n \, ds - \int_{\Omega_k} F \partial v^v \frac{\partial}{\partial x} \, dx = \int_{\Omega_k} v^v S \, dx
$$

(4)

where $\partial\Omega_k$ is the boundary of $\Omega_k$, and $v^v$ are test functions which are the same as the basis polynomials in Galerkin method. $F^R$ is the numerical flux at element boundary. Since the discontinuities are permitted at the interfaces of elements in the DG method, some numerical flux approximation should be adopted. Various numerical formulas for flux approximation have been proposed in the literature, and the references cited therein. Most of the approximations can be generalized as

$$
F^R = H(U_L, U_R, n)
$$

(5)

where $U_L$ and $U_R$ are the approximation values of $U$ on the left and the right side of the element edge, which is directed by the orientation of the normal vector of the edge pointing outside of the dominant element, $n$, according to the right-hand rule. For linear problems, the flux can be expressed as

$$
F^R = \tilde{A}_L U_L + \tilde{A}_R U_R,
$$

(6)

Let $\tilde{A} = \sum_{i=1}^{d} A_i n_i$. Then for the characteristics-based flux

$$
\tilde{A}_L = \frac{\tilde{A} + \theta|\tilde{A}|}{2}, \quad \tilde{A}_R = \frac{\tilde{A} - \theta|\tilde{A}|}{2}
$$

and for the Lax-Friedrich flux

$$
\tilde{A}_L = \frac{\tilde{A} + \theta \lambda_{max} I}{2}, \quad \tilde{A}_R = \frac{\tilde{A} - \theta \lambda_{max} I}{2}
$$

where $\lambda_{max}$ is the absolute maximum of the eigenvalues of $\tilde{A}$, and $\theta$ is an upwind-parameter that is usually chosen to be a positive real number between 0 and 1.

Introducing a proper local coordinate given by $\xi = \{\xi_i\}_{i=1,\ldots,d}$, the volume integration in equation (4) can be evaluated conveniently in the reference element $\hat{\Omega}_k$ using the Jacobian $J_k$, while the surface integral term is still computed in physical domain. Then the weak integral equation for time integration can be obtained as

$$
Q \frac{\partial C^k}{\partial t} + \frac{1}{J_k} \int_{\partial\Omega_k} v^v F^R \cdot n \, ds - \int_{\hat{\Omega}_k} \nabla v^v J_k^{-1} F d\xi = \int_{\hat{\Omega}_k} v^v S d\xi
$$

(7)

where $Q$ is mass matrix, and

$$
J_k = \frac{\partial (x_1, \ldots, x_d)}{\partial (\xi_1, \ldots, \xi_d)} \quad J_k = |J_k|
$$

In next section, we will discuss an application of Runge-Kutta integration to the solution of semi-discrete equation with nonuniform time-step sizes. What should be pointed out here is that, the developed method is not only valid for discontinuous Galerkin method but also, with proper modifications, applicable to other spatial discretization methods, such as the finite difference schemes.

### III. Nonuniform time-step Runge-Kutta Method

Most of high-order Runge-Kutta schemes frequently used for time integration in aeroacoustics computations, for instance, the classical Runge-Kutta schemes, the third order TVD Runge-Kutta scheme, and the optimized low-dissipation and low-dispersion Runge-Kutta (LDDRK) schemes, are developed with a uniform time step size. In order to increase the computational efficiency, linear coupling procedures for high-order Runge-Kutta time integration scheme with nonuniform time-step size will be developed in this section.
III.A. Linear Formulation

After a spatial discretization, such as the DG scheme discussed in the previous section, the semi-discrete equations for time integration can be modeled as

\[
\frac{\partial U}{\partial t} = F(U)
\]  \quad (8)

The general explicit \( p \)-stage Runge-Kutta time integration scheme with step \( \Delta t \) can be written as

\[
\begin{align*}
k_1 &= F(U^n) \\
k_2 &= F(U^n + \Delta t a_{21} k_1) \\
&\quad \vdots \\
k_p &= F(U^n + \Delta t [a_{p1} k_1 + a_{p2} k_2 + \cdots + a_{pp-1} k_{p-1}]) \\
U^{n+1} &= U^n + \Delta t \sum_{i=1}^{p} b_i k_i
\end{align*}
\]  \quad (9)

where \( U^n \) and \( U^{n+1} \) indicate the numerical solutions at time level \( t_n = n\Delta t \) and \( t_{n+1} = (n+1)\Delta t \) respectively. Here, \( k_i \) is the \( i \)-th stage value within one Runge-Kutta step.

We first consider a case of nonuniform grid as illustrated in Figure 1, which serves as a schematic of the interface between coarse and fine meshes. The fine mesh time-step \( \Delta t_f \) is taken to be half of the coarse mesh time-step \( \Delta t_c \), i.e.,

\[
\Delta t_c = 2\Delta t_f \quad \text{(10)}
\]

Figure 1. Sketch of a nonuniform mesh with nonuniform time-step.

Although a ratio of 1/2 could be a common choice at grid interfaces, efficiency considerations on grid partitioning may lead to a larger or smaller ratio. As we will see, the basic formulation presented below can be extended to cases with an arbitrary ratio of time-step sizes.

Under assumption (10), the coarse mesh solution will advance one step while the fine mesh solution advance two steps. Specifically, assuming both meshes have been at the same time-level at \( t_n = n\Delta t_c \), the solution on the coarse mesh is advanced forward one step to \( t_{n+1} = t_n + \Delta t_c \) by the integration formulas
respectively. For convenience, we will refer to time level coarse/fine element that are to be used in the time integration of a neighboring fine/coarse element: above can be summarized as follows, in which we will use an over tilde to denote the values computed in a coupling procedure should be employed to ensure correct communication of the neighboring elements with the coarse mesh are needed for spatial discretization operator on the fine mesh, and vice versa. A suitable elements. Consequently, for elements next to an interface of fine and coarse meshes, the stage values on operator \( F \)

\[
\begin{align*}
\mathbf{k}_{c,1} &= F(\mathbf{U}_c^n) \\
\mathbf{k}_{c,2} &= F(\mathbf{U}_c^n + \Delta t_c \mathbf{a}_{21} \mathbf{k}_{c,1}) \\
\quad \vdots \\
\mathbf{k}_{c,p} &= F(\mathbf{U}_c^n + \Delta t_c \sum_{i=1}^{p} b_i \mathbf{k}_{c,i}) \\
\mathbf{U}_{c}^{n+1} &= \mathbf{U}_c^n + \Delta t_c \sum_{i=1}^{p} b_i \mathbf{k}_{c,i}
\end{align*}
\]

(11)

For the solution on the fine mesh, the first time-step in the Runge-Kutta time integration with step \( \Delta t_f \) from \( t_n \) to \( t_n' = t_n + \Delta t_f \) is

\[
\begin{align*}
\mathbf{k}^{(1)}_{f,1} &= F(\mathbf{U}_f^n) \\
\mathbf{k}^{(1)}_{f,2} &= F(\mathbf{U}_f^n + \Delta t_f \mathbf{a}_{21} \mathbf{k}^{(1)}_{f,1}) \\
\quad \vdots \\
\mathbf{k}^{(1)}_{f,p} &= F(\mathbf{U}_f^n + \Delta t_f \sum_{i=1}^{p} b_i \mathbf{k}^{(1)}_{f,i}) \\
\mathbf{U}_{f}^{n+1/2} &= \mathbf{U}_f^n + \Delta t_f \sum_{i=1}^{p} b_i \mathbf{k}^{(1)}_{f,i}
\end{align*}
\]

(12)

And the second step in fine mesh from \( t_n' = t_n + \Delta t_f \) to \( t_{n+1} = t_n + 2\Delta t_f \), is carried out as

\[
\begin{align*}
\mathbf{k}^{(2)}_{f,1} &= F(\mathbf{U}_f^{n+1/2}) \\
\mathbf{k}^{(2)}_{f,2} &= F(\mathbf{U}_f^{n+1/2} + \Delta t_f \mathbf{a}_{21} \mathbf{k}^{(2)}_{f,1}) \\
\quad \vdots \\
\mathbf{k}^{(2)}_{f,p} &= F(\mathbf{U}_f^{n+1/2} + \Delta t_f \sum_{i=1}^{p} b_i \mathbf{k}^{(2)}_{f,i}) \\
\mathbf{U}_{f}^{n+1} &= \mathbf{U}_f^{n+1/2} + \Delta t_f \sum_{i=1}^{p} b_i \mathbf{k}^{(2)}_{f,i}
\end{align*}
\]

(13)

In the above, subscripts 'c' and 'f' have been used to denote the solutions in the coarse and fine meshes, respectively. For convenience, we will refer to time level \( t_n' = t_n + \Delta t_f \) as the intermediate time level, where only the solution on one mesh (e.g. on fine mesh) would be integrated, and the level \( t_{n+1} = t_n + 2\Delta t_f \) as the common time level, where the solutions on both fine and coarse meshes would be calculated. Since the numerical flux approximation is applied at the interface of neighboring elements, the spatial operator \( F(\mathbf{U}) \) appeared on the right hand side of equations (11)–(13) are dependent on values in adjacent elements. Consequently, for elements next to an interface of fine and coarse meshes, the stage values on the coarse mesh are needed for spatial discretization operator on the fine mesh, and vice versa. A suitable coupling procedure should be employed to ensure correct communication of the neighboring elements with different time step sizes. The relations that are necessary to implement a nonuniform time-step Runge-Kutta scheme described above can be summarized as follows, in which we will use an over tilde to denote the values computed in a coarse/fine element that are to be used in the time integration of a neighboring fine/coarse element:

1. For the first step in the coarse mesh and the first step in the fine mesh:

   \[ (1.1) \text{ Vector } \{ \mathbf{k}_{f,i} \}_{i=1,2,\cdots,p} \text{ should be computed from known } \{ \mathbf{k}^{(1)}_{f,i} \}_{i=1,2,\cdots,p} \text{ in fine mesh for use in time advancing in coarse mesh in (11);} \]

   \[ (1.2) \text{ Vector } \{ \mathbf{k}^{(1)}_{c,i} \}_{i=1,2,\cdots,p} \text{ should be computed from known } \{ \mathbf{k}_{c,i} \}_{i=1,2,\cdots,p} \text{ in coarse mesh for use in time advancing in fine mesh in (12).} \]

2. For the second step in the fine mesh:
For linear problems, relations (1.1), (1.2) and (2.1) can be established by the fact that stage values \( \{k_i\} \) in the Runge-Kutta scheme are directly related to the time derivatives of \( U \). An analysis of Runge-Kutta scheme (9) shows that

\[
\begin{bmatrix}
k_1 \\
k_2 \\
\vdots \\
k_p 
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & c_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & c_{p2} & c_{p3} & \cdots & c_{pp} 
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \Delta t & 0 & \cdots & 0 \\
0 & 0 & \Delta t^2 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \Delta t^{p-1} 
\end{bmatrix} \begin{bmatrix}
\frac{\partial U}{\partial x} \\
\frac{\partial^2 U}{\partial x^2} 
\end{bmatrix} t=t_n
\]

(14)

where \( c_{ij} \) are related to the Runge-Kutta coefficients \( a_{ij} \) by

\[
\begin{align*}
c_{22} &= a_{21} \\
c_{32} &= a_{31} + a_{32}, & c_{33} &= a_{32}a_{21} \\
c_{42} &= a_{41} + a_{42} + a_{43}, & c_{43} &= a_{43}a_{32} + a_{43}a_{31} + a_{43}a_{21}, \\
c_{44} &= a_{43}a_{32}a_{21}, & \vdots & \vdots & \vdots \\
c_{p2} &= a_{p1} + \cdots + a_{pp-1}, & c_{p3} &= a_{pp-1}a_{p-1}a_{p-2} + \cdots + a_{p2}a_{21}, \\
\vdots & \vdots & \vdots & \vdots \\
c_{pp} &= a_{pp-1}a_{p-1}a_{p-2} \cdots a_{21}
\end{align*}
\]

(15)

By (14), it is straightforward to get the relation between \( \{\tilde{f}_{f,i}\} \) and \( \{k_{f,i}^{(1)}\} \) in the fine elements neighboring the interface as

\[
\begin{bmatrix}
\tilde{f}_{f,1} \\
\tilde{f}_{f,2} \\
\vdots \\
\tilde{f}_{f,p} 
\end{bmatrix} = CP_{\Delta t_f} P^{-1}_{\Delta t_f} C^{-1} \begin{bmatrix}
k_{f,1}^{(1)} \\
k_{f,2}^{(1)} \\
\vdots \\
k_{f,p}^{(1)} 
\end{bmatrix} \equiv T_f \begin{bmatrix}
k_{f,1}^{(1)} \\
k_{f,2}^{(1)} \\
\vdots \\
k_{f,p}^{(1)} 
\end{bmatrix}
\]

(16)

and similarly, the relation between \( \{\tilde{c}_{c,i}^{(2)}\} \) and \( \{k_{c,i}^{(1)}\} \) in the coarse elements neighboring the interface is

\[
\begin{bmatrix}
\tilde{c}_{c,1}^{(1)} \\
\tilde{c}_{c,2}^{(1)} \\
\vdots \\
\tilde{c}_{c,p}^{(1)} 
\end{bmatrix} = CP_{\Delta t_c} P^{-1}_{\Delta t_c} C^{-1} \begin{bmatrix}
k_{c,1} \\
k_{c,2} \\
\vdots \\
k_{c,p} 
\end{bmatrix} \equiv T_c^{(1)} \begin{bmatrix}
k_{c,1} \\
k_{c,2} \\
\vdots \\
k_{c,p} 
\end{bmatrix}
\]

(17)

Obviously, the coupling matrix \( T_c^{(1)} \) is the inverse of \( T_f \). It is important to note that they are both lower-triangular matrices. These are the relations for (1.1) and (1.2) for linear problems.

Further, relations for (2.1) can be established by utilizing a Taylor series expansion of (9) at the intermediate time level. For linear cases, the Taylor series expansion of \( \{k_{c,i}^{(1)}\} \) at \( t_n' = t_n + \Delta t_f \) up to order \( p \) will be used. Specifically, we have
\[
\begin{bmatrix}
k_1 \\
k_2 \\
\vdots \\
k_p
\end{bmatrix}
\approx CP_{\Delta t_f}B_{\Delta t_f}
\]\n
where \(B_{\Delta t_f}\) is an upper-triangular matrix of the form

\[
B_{\Delta t_f} = \begin{bmatrix}
1 & b_{12}\Delta t_f & b_{13}\Delta t_f^2 & \cdots & b_{1p}\Delta t_f^{p-1} \\
0 & 1 & b_{23}\Delta t_f & \cdots & b_{2p}\Delta t_f^{p-2} \\
0 & 0 & 1 & \cdots & b_{3p}\Delta t_f^{p-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]  

(19)

When no optimization is applied, \(b_{ij}\) are just the coefficients of Taylor series expansion given below:

\[
b_{ij} = \frac{1}{(j - i)!}
\]

Using (13) and (18), \(\tilde{k}^{(2)}_{c,i}\) that is needed in the second step of the fine mesh can be calculated from the known \(k_{c,i}\) by the following equation:

\[
\begin{bmatrix}
\tilde{k}^{(2)}_{c,1} \\
\tilde{k}^{(2)}_{c,2} \\
\vdots \\
\tilde{k}^{(2)}_{c,p}
\end{bmatrix}
= CP_{\Delta t_f}B_{\Delta t_f}P_{\Delta t_c}^{-1}C^{-1}
\equiv T^{(2)}_e
\begin{bmatrix}
k_{c,1} \\
k_{c,2} \\
\vdots \\
k_{c,p}
\end{bmatrix}
\]  

(20)

which establishes the relation (2.a).

With the formulas for the coupling matrices \(T_f\), and \(T^{(s)}_e, s = 1, 2\), given above, the coupling procedures can be carried out as described in (11)–(13).

The formulation introduced above is derived for a general \(p\)-stage Runge-Kutta scheme. It can be applied readily to specific Runge-Kutta schemes. The difference in the extensions for different RK schemes lies only on coefficient matrix \(C\) in formula (13). In this sense, the formulation presented here is general for linear problems. More details on the extension to the schemes frequently used in CAA are given in Appendix A.

III.B. Application to arbitrary ratio of time-step size

In practice, considerations of grid partitioning and computational efficiency may result in a ratio of time-step sizes other than 1/2. The formulation presented above can be extended to a more general time-step ratio.

We first generalize the coupling formulas described in the previous subsection for a mesh interface between time step \(\Delta t_1\) and \(\Delta t_2\) of arbitrary ratio, as shown in Figure 2. To advance a time step \(\Delta t_1\) in mesh 1 from \(t_n\) to \(t_n + \Delta t_1\), the Runge-Kutta stage values on mesh 2 at time \(t = t_n\) is needed. Following the analysis in 3.1, the necessary Runge-Kutta stage values in mesh 2 to be used for mesh 1 can be readily obtained as

\[
[K_2]_{t=t_n} = CP_{\Delta t_1}B_{\Delta t_2}P_{\Delta t_2}^{-1}C^{-1}[K_2]_{t=t_{m-1}}
\]

(21)

where \([K_2]_{t=t_n}\) denotes the column vector of the necessary stage values in mesh 2 for use in mesh 1 at time level \(t = t_n\), computed from the known stage values at the nearest earlier time level \(t_{m-1}\) in the neighboring mesh of \(\Delta t_2\). Matrix \(B_{\Delta t_2}\) has the same definition as (19) and here

\[
\Delta \tau_2 = t_n - t_{m-1}
\]

(22)
Similarly, to advance a time step $\Delta t_2$ in mesh 2 from $t_m$ to $t_m + \Delta t_2$, the necessary RK stage values in mesh 1 to be used in mesh 2 is

$$[\tilde{K}_1]_{t=t_n} = CP_{\Delta t_2}B_{\Delta \tau_1}P^{-1}_{\Delta t_1}C^{-1}[K_1]_{t=t_n}$$

(23)

where

$$\Delta \tau_1 = t_m - t_n$$

(24)

For the coupling at a common time level where $\Delta \tau = 0$, matrix $B_{\Delta \tau}$ becomes the identity matrix, and (21) and (23) will be consistent with (16) and (17) in 3.1. Thus, the formulas (21) and (23) are generalizations of the coupling both at common time level and intermediate time level.

Using (21) and (23), the time integrations for grid points with different time steps of an arbitrary ratio can be carried out synchronously. As an example, the coupling procedures for time step ratio $2/3$ will be described in detail. The grid structure and time levels are illustrated in Figure 3 where

$$\Delta t_c = \frac{3}{2}\Delta t_f$$

(25)

Let $[K_c^{(s)}], s = 1, 2$ and $[K_f^{(s)}], s = 1, 2, 3$ be stage-value vectors at the $s$-th time level between two common ones, and $[\tilde{K}_f^{(s)}], s = 1, 2$ and $[\tilde{K}_c^{(s)}], s = 1, 2, 3$ be the vectors for coupling procedures. The necessary relations for implementation of a nonuniform time-step Runge-Kutta scheme shown in Figure 3 can be summarized as follows:

1. For the first step in the coarse mesh and the first step in the fine mesh:
   - (1.1) Vector $[\tilde{K}_f^{(1)}]$ should be computed from known $[K_f^{(1)}]$ in fine mesh for the time advancing in coarse mesh;
   - (1.2) Vector $[\tilde{K}_c^{(1)}]$ should be computed from known $[K_c^{(1)}]$ in coarse mesh for the time advancing in fine mesh.

2. For the second step in the fine mesh:
   - (2.1) Vector $[\tilde{K}_c^{(2)}]$ should be computed from known $[K_c^{(1)}]$ in coarse mesh for the time advancing in fine mesh.
3. For the second step in the coarse mesh:

(3.1) Vector $[\tilde{K}_f^{(2)}]$ should be computed from known $[K_f^{(2)}]$ in fine mesh for the time advancing in coarse mesh.

4. For the third step in the fine mesh:

(4.1) Vector $[\tilde{K}_c^{(3)}]$ should be computed from known $[K_c^{(2)}]$ in coarse mesh for the time advancing in fine mesh.

These relations can be realized by formulas given in (21) and (23). Specifically, we have

For (1.1):
$$[\tilde{K}_f^{(1)}] = CP_{\Delta t_c} P_{\Delta t_f}^{-1} C^{-1} [K_f^{(1)}] \equiv T_f^{(1)} [K_f^{(1)}]$$

(26)

For (1.2):
$$[\tilde{K}_c^{(1)}] = CP_{\Delta t_f} P_{\Delta t_c}^{-1} C^{-1} [K_c^{(1)}] \equiv T_c^{(1)} [K_c^{(1)}]$$

(27)

For (2.1):
$$[\tilde{K}_c^{(2)}] = CP_{\Delta t_c} B_{\Delta t_f} P_{\Delta t_c}^{-1} C^{-1} [K_c^{(1)}] \equiv T_c^{(2)} [K_c^{(1)}]$$

(28)

For (3.1):
$$[\tilde{K}_f^{(2)}] = CP_{\Delta t_c} B_{\Delta t_f} P_{\Delta t_c}^{-1} C^{-1} [K_f^{(1)}] \equiv T_f^{(2)} [K_f^{(1)}]$$

(29)

For (4.1):
$$[\tilde{K}_c^{(3)}] = CP_{\Delta t_c} B_{\Delta t_f} P_{\Delta t_c}^{-1} C^{-1} [K_c^{(2)}] \equiv T_c^{(3)} [K_c^{(2)}]$$

(30)

In practice, the coupling matrices can be stored and do not need to be computed at each time step. Moreover, the coupling is only carried out for elements next to the interfaces of nonuniform time steps, so the additional storage and computational time incurred due to coupling are limited.

III.C. Optimization with minimal dissipation and dispersion errors

Runge-Kutta schemes with low dissipation and low dispersion errors are preferred in CAA. In order to minimize the dissipation and dispersion errors in the coupling procedures presented previously, we study a possible optimization of the coefficient matrices at intermediate levels. The coefficients in the first row in formula (19), $\{b_{ij}\}_{i=2,3,\ldots,p}$ will be modified so that the numerical amplification factor at intermediate time level will be consistent with that of corresponding LDDRK scheme, which is already optimized to have the minimal dissipation and dispersion errors.
The case with integral time-step size ratio \( m : 1 \) is considered here for convenience, for which the coupling procedures are only carried out on the coarse mesh.

Consider the model equation \( \partial u / \partial t = -q \partial u / \partial x \). Upon Fourier expansion, the numerical amplification factor of Runge-Kutta integration with time step \( \Delta t_f \) can be expressed in the form of

\[
ro = 1 + \sum_{j=1}^{p} c_j (-i\sigma)^j
\]  

(31)

where \( \sigma = q k^* \Delta t_f \), and \( k^* \) is only dependant on spatial discretization; for LDDRK schemes \( c_j \) are the coefficients given in. After \( m \) steps advancing with uniform step size, the factor should be \( r_o^m \).

After applying coupling procedure, the amplification factor at an intermediate time level in coarse mesh becomes

\[
\tilde{r} = 1 + \sum_{j=1}^{p} \gamma_j (-i\sigma)^j
\]  

(32)

where \( \sigma \) has the same definition as before, and coefficients \( \{\gamma_j\}_{j=1,2,\cdots,p} \) are related to \( \{b_{ij}\}_{j=2,3,\cdots,p} \) by formula (18).

Thus the amplification factor of nonuniform time-step advancing with ratio \( m : 1 \) can be obtained at each common time level by

\[
\tilde{r}^{(k)} = \frac{ro}{\tilde{r}^{(k-1)}}
\]

\[
= \left(1 + \sum_{j=1}^{p} c_j (-i\sigma)^j\right)^{k-1} \left(1 + \sum_{j=1}^{p} \gamma_j (-i\sigma)^j\right), k = 2, 3, \cdots, m
\]  

(33)

where \( \tilde{r}^{(k)} \) is the factor of the coupling procedure at the \( k \)-th intermediate time level, of which the first one is \( r_o \). By comparing the factor \( \tilde{r}^{(k)} \) with that of LDDRK integration, \( r_o^{k} \), a linear system for \( b_{ij} \) is yielded.

The optimized coefficients turn out to be practically the same for all intermediate time levels (2nd level, 3rd level, \( \cdots \)) within one coarse time step of different integral time-step ratio. Table 1 shows the optimized coefficients for \( m = 2, 3, 4 \). Once the optimized coefficients \( \{b_{ij}\}_{j=2,3,\cdots,p} \) have been determined, the matrices \( B \) and \( T^{(2)} \) are to be found by formulas (18) and (20). The extension to the cases with arbitrary time-step size ratio are straightforward. In actual computation, however, since the coupling procedure affects only a few elements next to the interfaces of nonuniform meshes, the benefit of this optimization to the solution on the whole domain has been found to be moderate.

Table 1. Optimized Coefficients for LDDRK Schemes at Different Time Levels.

<table>
<thead>
<tr>
<th>Stages</th>
<th>Level</th>
<th>( b_{14} )</th>
<th>( b_{15} )</th>
<th>( b_{16} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>0.1629970000</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1657492500</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.1662589258</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.1665580000</td>
<td>0.03944976670</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1666304445</td>
<td>0.04126266474</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.1666545926</td>
<td>0.04157650988</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1/3!</td>
<td>0.041666666670</td>
<td>0.007810050010</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1/3!</td>
<td>0.04166666669</td>
<td>0.008300628140</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1/3!</td>
<td>0.04166666670</td>
<td>0.008326873060</td>
</tr>
</tbody>
</table>

Note. Common time level is the 1st level; \( b_{12} = 1, b_{13} = 1/2 \).
IV. Stability Analysis

In this section, we consider the stability of the coupling procedure proposed in previous sections. Specifically, we study the stability of computation for a nonuniform mesh shown in Figure 4 when the nonuniform Runge-Kutta scheme is applied to the one-dimensional wave equation

\[
\frac{\partial u}{\partial t} + q \frac{\partial u}{\partial x} = 0, \quad x \in \Omega, \ t \leq T
\]  

(34)

where \(u\) is the unknown, and \(q\) is the speed of the wave. We also assume periodic boundary condition in our stability analysis.

![Figure 4. Schematic of a finite nonuniform mesh composed of two blocks.](image)

In Figure 4, the total number of elements is assumed to be \(2N_e\), with \(N_e\) elements, indexed from 1 to \(N_e\), in the fine mesh region and \(N_e\) elements, indexed from \(-N_e + 1\) to 0, in the coarse mesh region. For interior elements, the semi-discrete DG approximatin can be written as

\[
d\mathbf{C}^k = \frac{1}{\Delta x_k} [\mathbf{M}^+ \mathbf{C}^k + \mathbf{M}^- \mathbf{C}^{k-1}] + \mathbf{M}_0 \mathbf{C}^k
\]  

(35)

for \(k = -N_e + 2, -N_e + 3, \ldots, N_e - 1\), where \(\mathbf{C}^k\) is a column vector that contains all expansion coefficients of solution in element with index \(k\), and \(\Delta x_k = \Delta x_c\) and \(\Delta x_f\) for the elements in coarse and fine meshes respectively. \(\mathbf{M}^0, \mathbf{M}^-\) and \(\mathbf{M}^+\) are the coefficient matrices for the dominant element and its left and right adjacent elements respectively, and more details are given in Appendix B. For the first and last element, due to periodic boundary condition, we have

\[
d\mathbf{C}^{-N_e+1} = \frac{1}{\Delta x_c} \left[ \mathbf{M}^- \mathbf{C}^{N_e-1} + \mathbf{M}^0 \mathbf{C}^{-N_e+1} + \mathbf{M}^+ \mathbf{C}^{N_e+1} \right]
\]  

(36)

and

\[
d\mathbf{C}^{N_e} = \frac{1}{\Delta x_f} \left[ \mathbf{M}^- \mathbf{C}^{N_e} + \mathbf{M}^0 \mathbf{C}^{N_e} + \mathbf{M}^+ \mathbf{C}^{-N_e+1} \right]
\]  

(37)

The above can be easily written in a block-matrix form as

\[
\frac{d\mathbf{C}}{dt} = \mathbf{C} \equiv \mathbf{HC}
\]  

(38)

11 of 36

American Institute of Aeronautics and Astronautics Paper 2009-3114
where $\tilde{C}$ is a column vector of length $N_g = N \cdot 2N_e$ containing all expansion coefficients on the whole grid, and $N$ is the degree of freedom in one element. Thus $\tilde{C}$ is of the form

$$\tilde{C} = \begin{bmatrix} C^{-N_e+1} \\ C^{-N_e+2} \\ \vdots \\ C^{N_e} \end{bmatrix}, \quad \text{with} \quad C^k = \begin{bmatrix} c^k_1 \\ c^k_2 \\ \vdots \\ c^k_{N_e} \end{bmatrix}, \quad k = -N_e+1, \ldots, N_e.$$

**IV.A. Analysis of RKDG with uniform time-step size**

We first present a study of stability analysis with a uniform time step. For the semi-discrete equation (36), the Runge-Kutta scheme can be expressed as

$$K_1 = \tilde{H} \tilde{C}^n$$

$$K_2 = \tilde{H}(\tilde{C}^n + \Delta t a_{21} K_1)$$

$$\vdots$$

$$K_p = \tilde{H}(\tilde{C}^n + \Delta t [a_{p1} K_1 + \cdots + a_{p(p-1)} K_{p-1}])$$

$$\tilde{C}^{n+1} = \tilde{C}^n + \Delta t [b_1 K_1 + b_2 K_2 + \cdots + b_p K_p]$$

From the above, it is straightforward to get a matrix relation

$$\tilde{C}^{n+1} = R \tilde{C}^n$$

Matrix $R$ will be referred to as the amplification matrix. The scheme would be stable if all eigenvalues of $R$ have a modulus less or equal to unity, i.e.,

$$\max(|\lambda_i|) \leq 1$$

where $|\lambda_i|$ is the modulus of the eigenvalue of $R$.

Obviously, the amplification matrix $R$ is dependent on the specific RK scheme, CFL number and the spacings of the coarse and fine element. Here we first present results for a uniform grid with $\Delta x_c = \Delta x_f = 1$ (Case 1) as well as a nonuniform grid with $\Delta x_c = 1, \Delta x_f = 1/2$ (Case 2) but with a single uniform time step. Figure 5[10] show eigenvalue distribution of the 4th-order DG scheme coupled with LDRRK4, LDDRK5 and LDDRK6, where the CFL number is at the stability limit. The CFL number is defined by the coarse mesh as

$$\text{CFL} = \frac{\Delta t_c}{\Delta x_c}$$

For LDDRK schemes and DG schemes up to $P = 6$, the stability limit on the CFL number determined by condition (45) has been found computationally and given in table 2 for case 1, for several LDDRK schemes and DG schemes. The stability limits obtained by direct numerical simulation are also listed. Very good agreement is observed.
Table 2. Stability limits for uniform LDDRK-DG with $\theta = 0.5$.

<table>
<thead>
<tr>
<th>Stage of LDDRK</th>
<th>Order of DG</th>
<th>CFL$_{max}$ Analysis</th>
<th>CFL$_{max}$ Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>2.29</td>
<td>2.29</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.646</td>
<td>0.646</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.329</td>
<td>0.329</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.202</td>
<td>0.202</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.138</td>
<td>0.137</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2.73</td>
<td>2.72</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.708</td>
<td>0.709</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.356</td>
<td>0.358</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.218</td>
<td>0.219</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.148</td>
<td>0.149</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.107</td>
<td>0.107</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>3.10</td>
<td>3.11</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.825</td>
<td>0.825</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.410</td>
<td>0.411</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.252</td>
<td>0.253</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.172</td>
<td>0.172</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.124</td>
<td>0.125</td>
</tr>
</tbody>
</table>
Figure 5. Distribution of Eigenvalues of the amplification matrix $R$ for uniform time-step LDDRK4-DG(O(3)): a) Case 1 when CFL = 0.329 and b) Case 2 when CFL = 0.1645.
Figure 6. Distribution of Eigenvalues of the amplification matrix $R$ for uniform time-step LDDRK4-DG($O(4)$): a) Case 1 when CFL = 0.202 and b) Case 2 when CFL = 0.101.
Figure 7. Distribution of Eigenvalues of the amplification matrix $R$ for uniform time-step LDDRK5-DG($\mathcal{O}(3)$): a) Case 1 when CFL = 0.356 and b) Case 2 when CFL = 0.178.
Figure 8. Distribution of Eigenvalues of the amplification matrix $R$ for uniform time-step LDDRK5-DG($\mathcal{O}(4)$): a) Case 1 when $\text{CFL} = 0.218$ and b) Case 2 when $\text{CFL} = 0.109$. 

(Units of magnitude are not specified in the figure, but are typically indicated in the context or caption.)
Figure 9. Distribution of Eigenvalues of the amplification matrix $R$ for uniform time-step LDDRK6-DG($O(3)$): a) Case 1 when CFL = 0.410 and b) Case 2 when CFL = 0.205.
Figure 10. Distribution of Eigenvalues of the amplification matrix R for uniform time-step LDDRK6-DG(\(O(4)\)): a) Case 1 when CFL = 0.252 and b) Case 2 when CFL = 0.126.
IV.B. Analysis of RKDG with nonuniform time-step size

We now consider the stability for the proposed nonuniform time-step RK schemes. By applying the coupled Runge-Kutta scheme (11)-(13) to the semi-discrete equation (36), together with coupling matrices developed in (16), (17) and (20), a matrix relation of the form (44) can again be obtained,

\[ \bar{C}^{n+1} = R(\Delta t_c, \Delta t_f) \bar{C}^n \]

where the amplification matrix \( R \) now depends on the time steps \( \Delta t_c \) and \( \Delta t_f \) of the coarse and fine meshes.

We consider the ratio \( \Delta t_c : \Delta t_f = \Delta x_c : \Delta x_f = 2 : 1 \) (Case 3), so that CFL number is uniform on the entire domain. Obviously, the eigenvalues of \( R(\Delta t_c, \Delta t_f) \) for a nonuniform time step will differ from those for a uniform time step considered in 4.1. We have tried to plot the eigenvalues of \( R \) for LDDRK-DG(\( O(1-6) \)) at the corresponding maximal CFL number obtained in the same way as in the previous section. For comparing with the results of Case 1 and Case 2, distributions of the eigenvalues of Case 3 that take non-uniformity of time-step size and element-spacing size into account are shown in Figure 11-13. Those maximal CFL numbers for nonuniform LDDRK-DG schemes are in a good agreement with the stability limits obtained in direct numerical simulation, given as in Table 3. It is observed that the stability limits in nonuniform time-step cases remain the same or are slightly larger than those of uniform LDDRK-DG schemes in Table 2, except when the order of DG scheme is higher than 3 and in combination with LDDRK5, where the stability limits of nonuniform time-step for LDDRK5-DG schemes drop down slightly. As such, use of coupling procedures is not expected to have a negative impact on the stability limit of RK schemes in general.

<table>
<thead>
<tr>
<th>Stage of order of</th>
<th>CFL(_{max})</th>
<th>Analysis</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>LDDRK DG</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2.31</td>
<td>2.31</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.659</td>
<td>0.659</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.334</td>
<td>0.334</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.204</td>
<td>0.204</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.138</td>
<td>0.135</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2.76</td>
<td>2.76</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.719</td>
<td>0.719</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.365</td>
<td>0.364</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.214</td>
<td>0.214</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.139</td>
<td>0.139</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.0982</td>
<td>0.0982</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>3.08</td>
<td>3.08</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.832</td>
<td>0.832</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.417</td>
<td>0.417</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.255</td>
<td>0.254</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.173</td>
<td>0.173</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.125</td>
<td>0.125</td>
</tr>
</tbody>
</table>

Note: upwind parameter \( \theta = 0.5 \).
V. Numerical examples

In this section, we present numerical examples of applying the nonuniform time step Runge-Kutta scheme developed in the previous sections. One dimensional test examples are shown in section 5.1 and two-dimensional examples are shown in section 5.2.

V.A. One-dimensional tests

V.A.1. Linear case

The first case is the one-dimensional scalar transport equation.

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad (48) \]

with the initial condition \( u(x, 0) = 0 \), on the interval \( x \in [0, 100] \). At the boundary \( x = 0 \), a sine wave \( u(0, t) = \sin(\pi x) \) is excited which propagates into the domain. We consider a nonuniform grid consisting of three blocks of which the second one is fine mesh and the other two are coarse meshes. The one-dimensional version of nonuniform time-step RKDG method is used to solve this problem. With a uniform CFL number, the time-step sizes of the neighboring blocks are different. Thus the coupling procedure should be adopted in the elements adjacent to the interfaces of fine and coarse meshes.

Fourth-order low-storage LDDRK scheme and corresponding coupling procedures are applied here combined with the same order DG method. The numerical solutions \( u_h \) are given when \( \Delta x_c : \Delta x_f = 3 : 2 \) in Figure 14 and \( \Delta x_c : \Delta x_f = 2 : 1 \) in Figure 15 respectively, where the grids are illustrated on the top of the graphs. In both examples, the sine wave propagates through the interfaces without any noticeable error.

The accuracy of the solution is further analyzed in a grid refinement study. We compare the numerical solution of the 20th period with the 1st period at \( t = 100 \) in the \( L^2 \)-norm of the error which is defined as

\[ E = \sqrt{\int_0^1 \left| u_h(x, t) - u_h(x + 20\lambda, t) \right|^2 dx} \]

(49)

The grid refinement results are shown in Table 4. The super-convergence rate is observed in the table. The linear coupling procedure involved in nonuniform time step Runge-Kutta integration preserves the super-convergence property of DG method.\(^{1,12}\)

<table>
<thead>
<tr>
<th>P</th>
<th>( \Delta x_c )</th>
<th>Error ( E )</th>
<th>Order</th>
<th>P</th>
<th>( \Delta x_c )</th>
<th>Error ( E )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.1337986E+01</td>
<td>3</td>
<td>1.0</td>
<td>0.2034853E-02</td>
<td>6</td>
<td>1.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1943643E+00</td>
<td>0.25</td>
<td>0.2946610E-06</td>
<td>6.7060</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.2802999E-01</td>
<td>0.125</td>
<td>0.3092206E-08</td>
<td>6.8989</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>0.3615612E-02</td>
<td>0.0625</td>
<td>0.8563485E-07</td>
<td>5.0159</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

V.A.2. Burger’s equation

In this example, the coupling procedure is applied to a nonlinear test problem. Consider one-dimensional Burger’s equation

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0 \]

with the initial condition \( u(x, 0) = 0 \), on the interval \( x \in [0, 100] \). At the boundary \( x = 0 \), a sine wave \( u(0, t) = \sin(\pi x) \) is excited which propagates into the domain. We consider a nonuniform grid consisting of three blocks of which the second one is fine mesh and the other two are coarse meshes. The one-dimensional version of nonuniform time-step RKDG method is used to solve this problem. With a uniform CFL number, the time-step sizes of the neighboring blocks are different. Thus the coupling procedure should be adopted in the elements adjacent to the interfaces of fine and coarse meshes.

Fourth-order low-storage LDDRK scheme and corresponding coupling procedures are applied here combined with the same order DG method. The numerical solutions \( u_h \) are given when \( \Delta x_c : \Delta x_f = 3 : 2 \) in Figure 14 and \( \Delta x_c : \Delta x_f = 2 : 1 \) in Figure 15 respectively, where the grids are illustrated on the top of the graphs. In both examples, the sine wave propagates through the interfaces without any noticeable error.

The accuracy of the solution is further analyzed in a grid refinement study. We compare the numerical solution of the 20th period with the 1st period at \( t = 100 \) in the \( L^2 \)-norm of the error which is defined as

\[ E = \sqrt{\int_0^1 \left| u_h(x, t) - u_h(x + 20\lambda, t) \right|^2 dx} \]

(49)

The grid refinement results are shown in Table 4. The super-convergence rate is observed in the table. The linear coupling procedure involved in nonuniform time step Runge-Kutta integration preserves the super-convergence property of DG method.\(^{1,12}\)

<table>
<thead>
<tr>
<th>P</th>
<th>( \Delta x_c )</th>
<th>Error ( E )</th>
<th>Order</th>
<th>P</th>
<th>( \Delta x_c )</th>
<th>Error ( E )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.1337986E+01</td>
<td>3</td>
<td>1.0</td>
<td>0.2034853E-02</td>
<td>6</td>
<td>1.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1943643E+00</td>
<td>0.25</td>
<td>0.2946610E-06</td>
<td>6.7060</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.2802999E-01</td>
<td>0.125</td>
<td>0.3092206E-08</td>
<td>6.8989</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>0.3615612E-02</td>
<td>0.0625</td>
<td>0.8563485E-07</td>
<td>5.0159</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

V.A.2. Burger’s equation

In this example, the coupling procedure is applied to a nonlinear test problem. Consider one-dimensional Burger’s equation

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0 \]

with the initial condition \( u(x, 0) = 0 \), on the interval \( x \in [0, 100] \). At the boundary \( x = 0 \), a sine wave \( u(0, t) = \sin(\pi x) \) is excited which propagates into the domain. We consider a nonuniform grid consisting of three blocks of which the second one is fine mesh and the other two are coarse meshes. The one-dimensional version of nonuniform time-step RKDG method is used to solve this problem. With a uniform CFL number, the time-step sizes of the neighboring blocks are different. Thus the coupling procedure should be adopted in the elements adjacent to the interfaces of fine and coarse meshes.

Fourth-order low-storage LDDRK scheme and corresponding coupling procedures are applied here combined with the same order DG method. The numerical solutions \( u_h \) are given when \( \Delta x_c : \Delta x_f = 3 : 2 \) in Figure 14 and \( \Delta x_c : \Delta x_f = 2 : 1 \) in Figure 15 respectively, where the grids are illustrated on the top of the graphs. In both examples, the sine wave propagates through the interfaces without any noticeable error.

The accuracy of the solution is further analyzed in a grid refinement study. We compare the numerical solution of the 20th period with the 1st period at \( t = 100 \) in the \( L^2 \)-norm of the error which is defined as

\[ E = \sqrt{\int_0^1 \left| u_h(x, t) - u_h(x + 20\lambda, t) \right|^2 dx} \]

(49)

The grid refinement results are shown in Table 4. The super-convergence rate is observed in the table. The linear coupling procedure involved in nonuniform time step Runge-Kutta integration preserves the super-convergence property of DG method.\(^{1,12}\)
\[ \frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} - \epsilon \frac{\partial^2 u}{\partial x^2} = 0 \] (50)

with a smooth initial condition \( u(x,0) = 1 - \tanh \left( \frac{x-x_0}{2\epsilon} \right) \), \( x_0 = 4 \), on the domain \( 0 \leq x \leq 50 \), and \( \epsilon = 0.2 \) is the viscous coefficient.

The nonuniform mesh \( \Delta x_c : \Delta x_f = 2 : 1 \) with the time-step ratio \( \Delta t_c : \Delta t_f = 2 : 1 \) is applied here. Fourth-order low-storage LDDRK scheme combined with fourth-order DG discretization is used. By introducing an auxiliary variable \( w = \partial u/\partial x \), equation (50) is rewritten into a system of two first-order equations. Because of this, the values of \( w \) are also required to be coupled properly at the interface of two meshes.

The numerical solutions and its first spatial derivative on nonuniform grid are compared with the exact solutions in Figure 16, which has the same shape as the initial condition as it propagates downstream. No oscillation is observed in the neighborhood of the interface of meshes at \( x = 25 \), which demonstrates the stability of the coupling procedures for a nonlinear case.

V.B. Two-dimensional test

For two-dimensional verification, the coupling procedure is applied to simulate the propagation of a Gaussian pulse, using the linearized Euler equations in the domain \( x \in [-100, 100] \), \( y \in [0, 200] \)

\[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial \mathbf{x}} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial \mathbf{y}} = 0 \] (51)

where
\[ \mathbf{U} = \begin{bmatrix} \rho' \\ u' \\ v' \\ p' \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u_0 & \rho_0 & 0 & 0 \\ 0 & u_0 & 0 & 1/\rho_0 \\ 0 & 0 & u_0 & 0 \\ 0 & \gamma p_0 & 0 & u_0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v_0 & 0 & \rho_0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & 1/\rho_0 \\ 0 & 0 & \gamma p_0 & v_0 \end{bmatrix} \]

The background flow parameters \( (\rho_0, u_0, v_0, p_0)^T \) are constant
\[ p_0 = 1/\gamma, \quad \rho_0 = 1, \quad u_0 = v_0 = 0 \] (52)

and the perturbing quantities \( (\rho', u', v', p')^T \) have the initial profile
\[ \rho' = p' = \exp\{-\ln 2 \left[ \frac{x^2 + (y-125)^2}{r_0^2} \right]\} \]
\[ u' = v' = 0, \quad r_0 = 10 \]

The linear coupling procedure is tested on the nonuniform triangular grid as shown as Figure 17, which consists of two blocks with different characteristic lengths of the elements that satisfy \( h_c : h_f = 2 : 1 \).

Contours of pressure at time \( t = 30, 45, 60, \) and 75 are given in Figure 18. The pulse propagates smoothly from the coarse mesh to the fine mesh. The comparisons of the numerical solutions with the exact one along the line \( x = 0 \) perpendicular to the interface of two blocks are shown in Figure 19 with excellent agreements.
Figure 11. Distribution of Eigenvalues of the amplification matrix $\bar{R}$ for nonuniform time-step LDDRK4-DG: a) LDDRK4-DG($O(3)$) when CFL = 0.334 and b) LDDRK4-DG($O(4)$) when CFL = 0.204.
Figure 12. Distribution of Eigenvalues of the amplification matrix $\bar{R}$ for nonuniform time-step LDDRK5-DG: a) LDDRK5-DG($O(3)$) when CFL = 0.365 and b) LDDRK5-DG($O(4)$) when CFL = 0.214.
Figure 13. Distribution of Eigenvalues of the amplification matrix $\bar{R}$ for nonuniform time-step LDDRK6-DG: a) LDDRK6-DG($O(3)$) when CFL = 0.417 and b) LDDRK6-DG($O(4)$) when CFL = 0.255.
Figure 14. Solution at a) $t = 40$, b) $t = 60$ and c) $t = 100$ when $\Delta t_1 : \Delta t_2 : \Delta t_3 = 3 : 2 : 3$. 

American Institute of Aeronautics and Astronautics Paper 2009-3114
Figure 15. Solution at a) $t = 40$, b) $t = 60$ and c) $t = 100$, when $\Delta t_1 : \Delta t_2 : \Delta t_3 = 2 : 1 : 2$; and d) the 1st and 20th periods at $t = 100$. 
Figure 16. Solutions $U$ and $\frac{\partial U}{\partial x}$ of Burger’s equation $t = 0, 16, 26,$ and $36$; the dashdotted line locates the interface of meshes.
Figure 17. 2D nonuniform grid: a) whole and b) local.
Figure 18. Contours of pressure at $t = 30, 45, 60$ and $75$ with $M_x = 0$, and $M_y = 0$
Figure 19. Pressure on the line $x = 0$ at $t = 30, 45, 60$ and 75
Figure 20. Pressure on the line $y = 125$ at $t = 30, 45, 60$ and 75
VI. Conclusions

For complex aeroacoustics problems, the application of Runge-Kutta discontinuous Galerkin method with a uniform time step often demands high computational consumption due to the constraint on time step afforded by the CFL condition based on the smallest elements. In this paper, high-order Runge-Kutta scheme with nonuniform time-step sizes is developed which makes the time-integration of semi-discrete systems on nonuniform grid more efficient. The linear formulation of coupling procedures that is valid for arbitrary ratio of time step sizes is derived based on the general $p$-stage Runge-Kutta formulas at first, and then extended readily to the Runge-Kutta schemes frequently used in CAA. The coupling procedure is only necessary for the elements neighboring the interface of two meshes advancing with different time-step sizes, thus the additional computational cost and storage caused by such coupling are limited in reality. An approach of optimization for the coupling coefficients is also carried out for low dissipation and low dispersion errors. Moreover, stability of the coupling procedure has also been examined for LDDRK schemes in combination with 1st to 6th order DG method. It was found that the coupling procedures for nonuniform time-step LDDRK-DG schemes generally preserve the stability property of the schemes with uniform time steps. For verification, numerical experiments have been performed on one-dimensional and two-dimensional linear problems. The numerical results illustrate the stability and accuracy properties of proposed coupling procedures. Furthermore, applicability of the linear procedure to nonlinear problem is also demonstrated by solving one-dimensional Burger’s equation. It has been pointed out that the nonuniform time-step RK method presented can be applied to not only DG method but also other typical spatial discretization method like finite difference schemes. The further application of the method will be reported in the future.

Appendix A: Extension to Runge-Kutta schemes frequently used in CAA

In this Appendix, we will show more details about the extension of the coupling procedures presented in Section 3 to TVD, $2N$-storage and minimum-storage Runge-Kutta schemes.

TVD Runge-Kutta Schemes

The third-order TVD Runge-Kutta time integration is used frequently for hyperbolic cases for the property of total variation diminishing and strong stability, especially the ones involving shocks. The general form of the scheme is as followed,

\[
U^{(0)} = U^n \\
U^{(i)} = (1 - \beta_i)U^{(0)} + \beta_i(U^{(i-1)} + \Delta tF(U^{(i-1)})) \\
U^{(n+1)} = U^m
\]

For third-order TVD Runge-Kutta scheme ($m = 3$),

\[
\beta_1 = 1, \quad \beta_2 = 1/4, \quad \beta_3 = 2/3
\]

To derive the coupling coefficient matrices, we rewrite it into the general $m$-stage Runge-Kutta form (9), and the corresponding coefficients are,

\[
a_{21} = 1, \quad a_{31} = 1/4, \quad a_{32} = 1/4 \\
b_1 = 1/6, \quad b_2 = 1/6, \quad b_3 = 2/3
\]

The coefficient matrices for coupling procedures can be obtained by formulae (14)-(20) in Section 3.

$2N$-Storage LDDRK Schemes

For efficient implementation of linear problems, LDDRK schemes are usually rewritten into special forms that require low storage, such as $2N$-storage schemes, where $N$ is the number of degree of freedom of the system.
2\(N\)-storage implementation is usually executed in the following way

\[
\begin{align*}
W_i &= \alpha_i W_{i-1} + h F(U_{i-1}) \\
U_i &= U_{i-1} + \beta_i W_i,
\end{align*}
\]  
(55)

where \(U_0 = U^{n-1}\) and \(U^n = U_p\); \(\alpha_i\) and \(\beta_i\) are the coefficients given in Ref. 24.

Then matrix \(C\) in coupling procedures is of the form

\[
\begin{pmatrix}
c_{11} & 0 & 0 & \cdots & 0 \\
c_{21} & c_{22} & 0 & \cdots & 0 \\
c_{31} & c_{32} & c_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{p1} & c_{p2} & c_{p3} & \cdots & c_{pp}
\end{pmatrix}
\]  
(56)

where

\[
\begin{align*}
c_{11} &= 1 \\
c_{21} &= 1 + \alpha_2 c_{11}, \\
c_{31} &= 1 + \alpha_3 c_{21}, \\
c_{33} &= \beta_3 c_{22}, \\
c_{41} &= 1 + \alpha_4 c_{31}, \\
c_{43} &= \beta_2 c_{22} + \beta_3 c_{32} + \alpha_4 c_{33}, \\
\cdots \\
c_{p1} &= 1 + \alpha_p c_{p-1}, \\
c_{p2} &= \beta_1 c_{11} + \beta_2 c_{21} + \cdots + \beta_{p-1} c_{p-1} + \alpha_p c_{p-11}, \\
\cdots \\
c_{pp} &= \beta_{p-1} c_{p-1p-1}
\end{align*}
\]  
(57)

Minimum Storage Runge-Kutta Schemes

Most recently, a new minimum storage Runge-Kutta scheme of fourth-order accuracy with six stages is proposed by M. Calvo and his co-workers in Ref. 5, which only require 2\(N\)-storage with low dissipation and low dispersion errors. This algorithm can be written equivalently as the \(p\)-stage Runge-Kutta schemes

\[
\begin{align*}
F_1 &= F(U^n), \\
F_i &= F(U^{n+1} + \Delta t \sum_{j=1}^{i-1} b_j F_j + \Delta t \gamma_{i-1} F_{i-1}), \quad (i = 2, \cdots, p) \\
U^{n+1} &= U^n + \Delta t \sum_{j=1}^{p} b_j F_j.
\end{align*}
\]  
(58)

Thus, the coupling matrix \(C\) has the same form as in formula (54), but the entries become

\[
\begin{align*}
c_{22} &= b_1 + \gamma_1 \\
c_{32} &= b_1 + b_2 + \gamma_2, \\
c_{33} &= (b_2 + \gamma_2)(b_1 + \gamma_1) \\
\cdots \\
c_{p2} &= \sum_{i=1}^{p} b_i + \gamma_{p-1}, \\
c_{p3} &= \sum_{i=2}^{p-1} b_i c_{i2} + \gamma_{p-1} c_{p-12}, \\
\cdots \\
c_{pp} &= \prod_{i=1}^{p-1}(b_i + \gamma_i)
\end{align*}
\]  
(59)

APPENDIX B: Matrices in the block-matrix form of DG discretization

In this section we give the matrices appear in Eq. (38).

For one dimensional test equation (34), the basis set is chosen to be \(\{v_i\}_{i=1}^{N}\). Utilizing the flux formula (33), matrices \(M\)s resulted in semi-discrete DG approximation as in (38) are defined as
\[ M^+ = qQ^{-1}[-\alpha B_{(1,-1)}] \]
\[ M^0 = qQ^{-1}[-\alpha^+ B_{(1,1)} + \alpha^- B_{(-1,-1)} + 2G] \]
\[ M^- = qQ^{-1}[\alpha^+ B_{(1,-1)}] \]

where \(Q\) is the mass matrix, and the matrices \(B_{(\alpha,\beta)}\) and \(G\) have the same definitions as in Ref. 12, of which the entries are defined as follows

\[
\begin{align*}
\{Q\}_{ij} &= \int_{-1}^{1} v_i v_j d\xi, \\
\{B_{(\alpha,\beta)}\}_{ij} &= v_i(\alpha) v_j(\beta), \\
\{G\}_{ij} &= \int_{1}^{1} \frac{\partial v_i}{\partial \xi} v_j d\xi, \quad i, j = 1, 2, \cdots, N.
\end{align*}
\]  

Acknowledgments

This work is supported by grants from the National Natural Science Foundation of China 50890181 and 50676003, the 973 Program-2007CB714604 (Li Liu and Xiaodong Li) and the National Science Foundation DMS-0810946 (Fang Q. Hu). Li Liu would like to acknowledge the support from China Scholarship Council which renders her stay at the Old Dominion University through 2007-2008. The authors also would like to acknowledge the 111 Project B07009 of China.

References


