PML Absorbing Boundary Condition for Nonlinear Euler Equations in Primitive Variables

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Absorbing Boundary Condition for Nonlinear Euler Equations in primitive variables are presented based on the Perfectly Matched Layer (PML) technique. To facilitate the application of PML complex change of variables in the frequency domain, new auxiliary variables are introduced. The final form of the absorbing equations is presented in unsplit physical primitive variables. Both the two-dimensional Cartesian and axisymmetric cases are considered. Although the nonlinear absorbing equations are not theoretically perfectly matched, numerical experiments show satisfactory results. The derived equations are tested in numerical examples and compared with the PML absorbing boundary condition in conservation form that was formulated in an earlier work. No significant difference in performance is found for the two formulations. A comparison with the linear PML in nonlinear problems is also considered. It is found that using nonlinear PML significantly improves the performance of the absorbing boundary condition.

I. Introduction

Perfectly Matched Layer (PML) is a technique of developing non-reflecting absorbing boundary conditions. In recent years, many progresses have been made in the development of PML for Computational Fluid Dynamics (CFD) and Computational Aeroacoustics (CAA), from linearized Euler equations to the fully nonlinear Euler and Navier-Stokes equations. In our previous study, the PML for nonlinear Euler and Navier-Stokes equations was given in the conservation form. In this paper, a PML for the nonlinear Euler equations in the primitive variables is developed. This is motivated by the observation that the nonlinear Euler equations in primitive variables remain a popular form of the governing equations for inviscid compressible flows, especially for many nonlinear aeroacoustics problems.

The purpose of this paper is two-fold. First, new absorbing boundary conditions in the primitive variables based on the PML technique are proposed, for both the two-dimensional and axisymmetric cases. Second, comparisons in the performance with the PML in the conservation form are conducted. In order to deal with the nonlinear terms involving spatial derivatives and to facilitate the application of PML complex change of variables in the frequency domain, new auxiliary variables are introduced. The final form of the absorbing equations is presented in unsplit physical primitive variables. Numerical examples are presented to demonstrate the validity and efficiency of the proposed boundary conditions.

In the next section, a time-domain PML for the nonlinear Euler equations in primitive variables is derived. It follows a three-step method. Firstly, a space-time transformation is applied to the governing equations. Secondly, a PML complex change of variables is applied in the frequency domain. Then the time domain PML absorbing boundary condition is derived by a conversion of the frequency domain equation. Numerical examples that validate the efficiency and validity of the proposed PML equations are presented in Section III. Concluding remarks are given in Section IV.

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II. Formulation of PML equations in primitive variables

The two-dimensional nonlinear Euler equations written in primitive variables are

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} &= 0 \\
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \gamma p \frac{\partial u}{\partial x} + \gamma p \frac{\partial v}{\partial y} &= 0
\end{align*}
\]  

(1) (2) (3) (4)

where \( u \) and \( v \) are the velocity components in \( x \) and \( y \) directions, respectively, \( p \) is the pressure, \( \rho \) is the density. In this paper, the velocity is nondimensionalized by a reference speed of sound \( c_\infty \), density by a reference density \( \rho_\infty \) and pressure by \( \rho_\infty c_\infty^2 \).

We will consider the construction of PML equations in the \( x \) and \( y \) directions. As shown in Figure 1, PML domains are introduced to absorb out-going disturbances.

We wish to formulate absorbing equations so that out-going disturbances can be exponentially reduced once they enter the PML domains while causing as little numerical reflection as possible. For nonlinear Euler equations, the solutions can be divided into two parts. One part is the time-independent mean state, the other part is the time-dependent fluctuation. It will be efficient to absorb only the time-dependent fluctuations. When the mean flow is unknown, a pseudo mean flow can be used in the formulation. Therefore, we express the primitive variables in the PML domain as

\[
u = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} = \begin{bmatrix} \bar{\rho} \\ \bar{u} \\ \bar{v} \\ \bar{p} \end{bmatrix} + \begin{bmatrix} \rho' \\ u' \\ v' \\ p' \end{bmatrix} = \bar{\nu} + \nu'
\] (5)

where the superscript “bar” indicates the mean flow and the superscript “prime” indicates the difference between the mean flow and the exact flow. The mean flow or pseudo mean flow should satisfy the steady
Euler equations,
\[
\begin{align*}
\bar{u} \frac{\partial \rho}{\partial x} + \bar{v} \frac{\partial \rho}{\partial y} + \rho \frac{\partial \bar{u}}{\partial x} + \bar{\rho} \frac{\partial \bar{v}}{\partial y} &= 0 \\
\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} &= 0 \\
\bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} &= 0 \\
\bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{v} \frac{\partial \bar{p}}{\partial y} + \gamma \bar{\rho} \frac{\partial \bar{u}}{\partial x} + \gamma \bar{\rho} \frac{\partial \bar{v}}{\partial y} &= 0
\end{align*}
\]

To facilitate the derivation of PML equations for (1)-(4), define new variables \( E \) and \( F \) as
\[
E = \frac{\partial u}{\partial x} = \begin{bmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial p}{\partial x} \end{bmatrix}, \quad F = \frac{\partial u}{\partial y} = \begin{bmatrix} \frac{\partial \rho}{\partial y} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial p}{\partial y} \end{bmatrix}
\]

And according to (5), we have
\[
E = \bar{E} + E', \quad F = \bar{F} + F'
\]

where
\[
E' = \frac{\partial(u - \bar{u})}{\partial x}, \quad F' = \frac{\partial(u - \bar{u})}{\partial y}
\]

Substitute (10) to the Euler equations, (1)-(4) can be rewritten as
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \bar{E}_\rho + v F_\rho + \rho E_u + \rho pF_v &= 0 \\
\frac{\partial u}{\partial t} + u \bar{E}_u + v F_u + \frac{1}{\rho} E_p &= 0 \\
\frac{\partial v}{\partial t} + u \bar{E}_v + v F_v + \frac{1}{\rho} F_p &= 0 \\
\frac{\partial p}{\partial t} + u \bar{E}_p + v F_p + \gamma pE_u + \gamma pF_v &= 0
\end{align*}
\]

where \( E \) and \( F \) will be treated as independent variables in our derivation below. In particular, we note that there are no spatial derivatives present in the above set of equations. After subtracting out the mean flow, we have
\[
\begin{align*}
\frac{\partial \rho'}{\partial t} + u \bar{E}_\rho - \bar{\rho} \bar{E}_\rho + v F_\rho - \bar{\rho} \bar{F}_\rho + \rho E_u - \bar{\rho} E_u + \rho pF_v - \bar{\rho} F_v &= 0 \\
\frac{\partial u'}{\partial t} + u \bar{E}_u - \bar{\rho} \bar{E}_u + v \bar{F}_u - \bar{\rho} \bar{F}_u + \frac{1}{\rho} E_p - \frac{1}{\bar{\rho}} \bar{E}_p &= 0 \\
\frac{\partial v'}{\partial t} + u \bar{E}_v - \bar{\rho} \bar{E}_v + v \bar{F}_v - \bar{\rho} \bar{F}_v + \frac{1}{\rho} F_p - \frac{1}{\bar{\rho}} \bar{F}_p &= 0 \\
\frac{\partial p'}{\partial t} + u \bar{E}_p - \bar{\rho} \bar{E}_p + v \bar{F}_p - \bar{\rho} \bar{F}_p + \gamma pE_u - \gamma \bar{\rho} E_u + \gamma pF_v - \gamma \bar{\rho} F_v &= 0
\end{align*}
\]

We shall derive the equations that absorb \( u', \ E', \ F' \).

For the stability of PML, a space-time transformation of the form
\[
t = t + \beta x
\]
functions of

Finally, we have the PML equations in primitive variables as follows

where

Now to rewrite (28) and (29) in the time domain, define

In the unsplit approach, multiplying 1 + \( \frac{i\sigma_x}{\omega} \) and 1 + \( \frac{i\sigma_y}{\omega} \) to (26) and (27), respectively, we have

In the frequency domain, where \( \sigma_x \) and \( \sigma_y \) are absorption coefficients, which are positive and could be functions of \( x \) and \( y \), respectively.\(^7\) After applying (24) and (25) to (23), it yields the following:

In the unsplit approach, multiplying 1 + \( \frac{i\sigma_x}{\omega} \) and 1 + \( \frac{i\sigma_y}{\omega} \) to (26) and (27), respectively, we have

Now to rewrite (28) and (29) in the time domain, define

where \( \textbf{q}_1 \) and \( \textbf{q}_2 \) are auxiliary variables. It is easy to find that the equations for auxiliary variables \( \textbf{q}_1 \) and \( \textbf{q}_2 \) are

Finally, we have the PML equations in primitive variables as follows

\[ \frac{\partial \rho}{\partial t} + u E_p - \bar{u} E_p + v F_p - \bar{v} F_p - \rho E_u - \bar{\rho} E_u + \rho F_v - \bar{\rho} F_v = 0 \]  
\[ \frac{\partial u}{\partial t} + u E_u - \bar{u} E_u + v F_u - \bar{v} F_u + \frac{1}{\rho} E_p - \frac{1}{\bar{\rho}} E_p = 0 \]  
\[ \frac{\partial v}{\partial t} + u E_v - \bar{u} E_v + v F_v - \bar{v} F_v + \frac{1}{\rho} F_p - \frac{1}{\bar{\rho}} F_p = 0 \]  
\[ \frac{\partial p}{\partial t} + u E_p - \bar{u} E_p + v F_p - \bar{v} F_p + \gamma p E_u - \gamma \bar{\rho} E_u + \gamma p F_v - \gamma \bar{\rho} F_v = 0 \]  

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in which \( \mathbf{E} \) and \( \mathbf{F} \) used in the above are computed as

\[
\mathbf{E} = \frac{\partial \mathbf{u}}{\partial x} - \sigma_x \mathbf{q}_1 + \beta \sigma_x (\mathbf{u} - \bar{\mathbf{u}})
\]

\[
\mathbf{F} = \frac{\partial \mathbf{u}}{\partial y} - \sigma_y \mathbf{q}_2
\]

and \( \mathbf{q}_1 \) and \( \mathbf{q}_2 \) are computed by (31) and (32).

\( \mathbf{E} \) and \( \mathbf{F} \) can be eliminated by directly substituting (37) and (38) into (33)-(36), and we get

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho}{\partial x} - \frac{\partial \bar{\rho}}{\partial x} - \frac{\partial \bar{\rho}}{\partial y} = 0
\]

\[
- \sigma_x [u q_{1,1} + \rho q_{1,0}] + \beta \sigma_x [u (\rho - \bar{\rho}) + \rho (u - \bar{u})] - \sigma_y [v q_{2,0} + \rho q_{2,1}] = 0
\]

Then, we can write PML equations (39)-(42) as

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} = 0
\]

where

\[
\mathbf{u} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/\rho \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \gamma p & v \end{bmatrix}
\]

Then, we can write PML equations (39)-(42) as

\[
\frac{\partial u}{\partial t} + A \left[ \frac{\partial u}{\partial x} + \beta \sigma_x (u - \bar{u}) - \sigma_x \mathbf{q}_1 \right] + B \left[ \frac{\partial u}{\partial y} - \sigma_y \mathbf{q}_2 \right] - \bar{A} \frac{\partial \mathbf{u}}{\partial x} - \bar{B} \frac{\partial \mathbf{u}}{\partial y} = 0
\]

where

\[
\mathbf{q}_1 = \begin{bmatrix} q_{1,0} \\ q_{1,1} \\ q_{1,v} \\ q_{1,p} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} q_{2,0} \\ q_{2,1} \\ q_{2,v} \\ q_{2,p} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \bar{u} \\ \bar{\rho} \\ \bar{\varepsilon} \\ \bar{\rho} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \bar{u} & \bar{\rho} & 0 & 0 \\ 0 & \bar{u} & 0 & 1/\bar{\rho} \\ 0 & 0 & \bar{u} & 0 \\ 0 & \gamma \bar{\rho} & 0 & \bar{u} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \bar{v} & 0 & \bar{\rho} & 0 \\ 0 & \bar{v} & 0 & 0 \\ 0 & 0 & \bar{v} & 1/\bar{\rho} \\ 0 & 0 & \gamma \bar{\rho} & \bar{v} \end{bmatrix}
\]

Equations (44), (31) and (32) are to be solved in the PML domain.

### III. Numerical examples

In this section, we present numerical examples of using PML absorbing boundary conditions in primitive variables for nonlinear Euler equations. In all the examples, the Dispersion-Relative-Preserving (DRP) scheme\(^8\) is applied for spatial discretization and the optimized 5- and 6-stage alternating Low-Dissipation and Low-Dispersion Runge-Kutta (LDDRK) scheme\(^9\) is used for time integration. Numerical examples include an isentropic vortex propagating downstream, a pressure wave propagation, roll-up vortices of a shear flow and nonlinear acoustic radiation from an oscillating piston in a wall.
III.A. Isentropic vortex

This numerical example is a classical test case for the performance of nonlinear non-reflecting boundary condition. A similar calculation has been performed for the PML absorbing boundary condition of the nonlinear Euler equations in conservation form in ref. A comparison in the performances of two formulations will be given here. The two-dimensional nonlinear Euler equations support an advective solution of the form

\[
\begin{pmatrix}
\rho(x, t) \\
u(x, t) \\
v(x, t) \\
p(x, t)
\end{pmatrix} = \begin{pmatrix}
0 \\
U_0 \\
V_0 \\
0
\end{pmatrix} + \begin{pmatrix}
\rho_r(r) \\
-u_r(r) \sin \theta \\
u_r(r) \cos \theta \\
p_r(r)
\end{pmatrix}
\] (45)

where \( r = \sqrt{(x-U_0 t)^2 + (y-V_0 t)^2} \), and for any given \( u_r(r) \) and \( \rho_r(r) \), the pressure \( p_r(r) \) is found by

\[
\frac{d}{dt} p_r(r) = \rho_r(r) u_r(r) \frac{r'}{r}
\] (46)

Equation (45) gives a solution that advects with constant velocity \((U_0, V_0)\). For our numerical tests, we consider a velocity distribution of the form

\[
u_r(r) = \frac{U'_{max}}{b} r e^{\frac{1}{2} \left(1 - \frac{r}{b} \right)}
\] (47)

where \( U'_{max} \) is the maximum velocity at \( r = b \). For an isentropic flow, we assume

\[
\rho_r = \frac{1}{\gamma} \rho_r^n
\] (48)

and, by integrating (46), we get the following density and pressure distributions,

\[
\rho_r(r) = \left(1 - \frac{1}{2}(\gamma - 1)(\frac{1}{U'_{max}})^2 e^{1 - \frac{r^2}{b^2}}\right)^{1/(\gamma - 1)}
\] (49)

\[
p_r(r) = \frac{1}{\gamma} \left(1 - \frac{1}{2}(\gamma - 1)(\frac{1}{U'_{max}})^2 e^{1 - \frac{r^2}{b^2}}\right)^{\gamma/(\gamma - 1)}
\] (50)

The initial condition is that given in (45) with \((U_0, V_0) = (0.5, 0), U'_{max} = 0.5U_0 = 0.25, \) and \( b = 0.2 \). The nonlinear Euler equation is solved by a finite difference computational aeroacoustics approach in a computational domain of \([-1.2, 1.2] \times [-1.2, 1.2] \) with a uniform grid of \( \Delta x = \Delta y = 0.02 \), including the surrounding PML domain of 10 grid points in width. In particular, the PML absorption coefficient is

\[
\sigma_x = \sigma_{max} \left| \frac{x-x_0}{D} \right| ^\alpha
\]

with \( \sigma_{max} = 20, \alpha = 4 \) and a similar model for \( \sigma_y \) is used. A grid stretching in the PML domain is also used to increase the efficiency of the absorbing zone. The stretching factor is

\[
\alpha(x) = 1 + 2 \left| \frac{x-x_0}{D} \right| ^2
\]

as noted in ref. The mean flow is naturally taken to be the same as the uniform background flow with parameter \( \beta = U_0/(1 - U_0^2) \).

Figure 2 shows instantaneous \( v \)-velocity contours at different times. It can be clearly observed that the vortex is absorbed effectively by the proposed PML boundary conditions.

To further assess the magnitude of reflection error, Figure 3 shows the time history of the maximum reflection error between the numerical solution and a reference solution obtained using a larger computational domain, along a vertical line \( x = 0.9 \) near the outflow boundary. The reflection errors are small and decrease with an increase in the width of the PML domain employed. Figure 4 plots the trend in the reduction of the maximum reflection error for the \( v \)-velocity component, as the PML width increases. Figure 4 also shows
Figure 2. $v$-velocity contour levels from $\pm 0.02$ to $\pm 0.24$. 
Figure 3. Maximum reflection error ($v$-velocity component) along $x = 0.9$ near the outflow boundary. PML width is as indicated. $U'_{\text{max}} = 0.5U_0$.

Figure 4. Maximum reflection error for the $v$-velocity component as a function of PML width $D$, $U'_{\text{max}} = 0.5U_0$. Solid line with squares: in primitive variables; dashed line with circles: in conservation form.
the results calculated by solving the PML equation in the conservation form. Maximum reflection errors are in the same order of magnitude. But, with the increase of the PML width, the reduction rate of the maximum reflection error calculated by solving the PML equation in primitive variables is slightly less than the one calculated by applying the PML equation in the conservation form.

Figure 5 plots the maximum reflection error in \( v \)-velocity component relative to the maximum velocity \( U_{max} \) of the vortex along \( x = 0.9 \) near the outflow boundary for various strengths of the vortex. The maximum relative reflection error is around 1% with PML width of 20 grid points when the vortex strength \( U_{max} = 0.8U_0 \). Figure 6 shows the maximum reflection error in a log scale, for the \( v \)-velocity component, as a function of the vortex strength \( U'_{max}/U_0 \). It also plots the results calculated by applying the PML in the conservation form. The performance of the PML equation in the conservation form is slightly better than the one in primitive variables.

III.B. Pressure pulse

In the second example, a strong pressure pulse is introduced into an uniform flow. The propagation of the pressure is simulated by applying the PML absorbing boundary condition for nonlinear Euler equations in primitive variables. The initial condition is

\[
\begin{pmatrix}
\rho \\
u \\
p
\end{pmatrix} =
\begin{pmatrix}
\rho_0 \\
u_0 \\
p_0
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} +
P'_{max}e^{-\ln(2)(x^2+y^2)/0.2^2}
\]

where \( \rho_0 = 1, U_0 = 0.5, V_0 = 0, P_0 = \frac{1}{\gamma} \) and \( P'_{max} = P_0 \). The computational domain and mesh are the same as those in the previous example. In both examples shown above, the pseudo mean flow is the background uniform flow, i.e. \( \rho = \rho_0 = 1, u = U_0 = 0.5, v = V_0 = 0 \) and \( p = P_0 = 1/\gamma \). Parameter \( \beta = \frac{U_0}{\gamma L_0^2} \).

Figure 7 presents the pressure contours at time \( t = 0, 0.5, 1.0 \) and 1.5, showing very little reflection by the PML domain. The pressure along \( y = 0 \) is plotted in Figure 8. A reference solution obtained using a large
Figure 6. Maximum reflection error for the $v$-velocity component as a function of vortex strength $U'_{\text{max}}/U_0$. PML width $D = 20 \Delta x$. Solid line with squares: in primitive variables; dashed line with circles: in conservation form.

computational domain is also plotted in dashed lines. Excellent agreement is found between the numerical and reference solutions in the physical domain.

To further assess the magnitude of reflection error, Figure 9 plots the time history of the maximum difference between the numerical and reference solutions for $P'_{\text{max}} = 1.0 P_0$, along lines $x = \pm 0.9$ and $y = \pm 0.9$ close to PML boundaries. The reflection errors are also quite small and decrease with an increase in the width of the PML domain employed. The maximum reflection errors for the pressure using the linear PML in primitive variables given in ref. are also plotted in Figure 9. We can see that for this strongly nonlinear problem, maximum reflection errors for the pressure are almost close to 9% and they do not decrease with an increase in the width of the PML domain employed.

Maximum reflection errors for the pressure relative to the maximum pressure $P'_{\text{max}}$ close to PML domains are plotted in Figure 10. The relative error is less than 0.2% for all cases with PML width of 20 grid points, although reflection error increases with the strength of the pressure. Figure 11 shows the maximum reflection error for the pressure as a function of the pressure strength $P'_{\text{max}}/P_0$. The results computed by applying the PML in the conservation form are also given in the figure. The performance of the PML equation in the conservation form is slightly better than the one in primitive variables when the strength of pressure is strong. But if pressure strength becomes very weak, the PML equation in primitive variables shows slightly better performance.

III.C. Roll-up vortices

In this numerical example, the same computation of roll-up vortices as that studied in ref are carried out by applying the PML equations in primitive variables. The initial condition in primitive variables is

$$
\begin{pmatrix}
\rho \\
u \\
0 \\
p
\end{pmatrix} =
\begin{pmatrix}
\rho_0(y) \\
U_0(y) \\
0 \\
\frac{1}{\gamma}
\end{pmatrix}
$$

(52)
and develop into roll-up vortices. The entire computational domain is \( \omega \) where a source term is added to Equation (4) to induce the instability wave. The source term is of the form

\[
A \frac{\partial T}{\partial t} + \frac{\partial}{\partial x} \left( \bar{u} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \bar{v} \frac{\partial T}{\partial y} \right) + \gamma - \frac{1}{2} \frac{\partial^2 T}{\partial x^2} = \frac{1}{T_0(y)}
\]

with

\[
T_0(y) = T_1 \frac{U_0 - U_2}{U_1 - U_0} + T_2 \frac{U_1 - U_0}{U_1 - U_2} + \gamma - \frac{1}{2} \frac{(U_1 - U_0)(U_0 - U_2)}{T_0(y)}
\]

where the mean temperature \( T_0(y) \) is determined by the Crocco relation for compressible flows. The parameters are

\[
U_1 = 0.8, \quad U_2 = 0.2, \quad \delta = 0.4, \quad T_1 = 1, \quad T_2 = 0.8, \quad \gamma = 1.4.
\]

A source term is added to Equation (4) to induce the instability wave. The source term is of the form

\[
s(x, y, t) = \sin(\omega t) e^{-\frac{(x-x_0)^2+(y-y_0)^2}{r_0^2}}
\]

where \( \omega = \pi/2, \quad (x_0, y_0) = (-0.5,0) \) and \( r_0 = 0.03 \).

The added source term excites the Kelvin-Helmholtz instability wave which would grow exponentially and develop into roll-up vortices. The entire computational domain is \([-1.5, 9.5] \times [-1.1, 1.1]\) with \( \Delta x = 0.05 \) and \( \Delta y = 0.01 \). Nonlinear Euler equations (43) are solved in the interior domain, and PML equations (44), (31) and (32) are solved in the surrounding PML domain with a width of 10 grid points.

Figure 12 shows vorticity contours in progressive time frames. The out-going vorticities are absorbed exponentially going through the PML domain. Figure 13 plots the time history of the pressure and \( \bar{v} \)-velocity at point \((x, y) = (8.75,0)\) close to the outflow boundary. The numerical (Solid line) and reference (circles) solutions are in good agreement.

For the calculation shown in Figure 12 and Figure 13, the pseudo mean flow \( \bar{u} \) is the same as the initial condition (52). And parameter \( \beta = 1/1.4 \) according to the linear wave analysis.

III.D. Acoustic radiation from an oscillating piston

III.D.1. Two-dimensional coordinate system

In this example, acoustic radiation from an oscillating piston in a wall is simulated. The Euler equations in primitive variables are solved in the physical domain and the PML equations (39)-(42) are solved in PML domain, as shown in Figure 14. The wall is at \( y = 0 \). The piston is located at \(-10 \leq x \leq 10, \quad y = 0\). And velocity of the piston \( v = A \sin(\pi t/20) \), where \( A \) is a constant. The entire computational domain is \( 0 \leq y \leq 105, \quad -105 \leq x \leq 105 \) with \( \Delta x = \Delta y = 0.5 \). Two-dimensional nonlinear PML equations (44), (31) and (32) are solved in PML domains with a width of 10 grid points at the top, left and right boundaries. And parameter \( \beta \) is set based on the mean flow \( U_0 \), i.e., \( \beta = U_0/(1 - U_0^2) \). Different cases will be simulated, including different velocity amplitudes of the piston \( A \) and different incoming flows \( U_0 \).

Figure 15 shows pressure contours at the beginning of a period of piston oscillation for \( A = 0.2 \) with different incoming flow \( U_0 \). Dashed lines indicate reference solutions calculated by a large computational domain of \([-320, 320] \times [0, 320]\). Figure 16 plots the pressure along line \( x = 0 \) over a period of piston oscillation for \( U_0 = 0.5 \), from top to bottom. The reference solution is also plotted in dashed lines. In Figure 17, we show the time history of the maximum reflection error for the pressure along lines \( x = \pm 97.5 \) and \( y = 97.5 \) close to the PML domain, from \( t = 0 \) to \( t = 500 \). The reflection error is obtained by comparing numerical solution with the reference solution. It is interesting to note that, while the reflection error increases with the amplitude of the nonlinear waves, it is largely independent of the mean flow.

Figure 18 shows pressure contours calculated by applying the linear PML\(^1\) for \( U_0 = 0.2 \). The reference solution computed using a large computational domain is also plotted in Figure 18. In this graph, reflection errors for the pressure close to the open boundaries are easily visible. The maximum reflection errors for the pressure using the linear and nonlinear PML in primitive variables, are compared in Figure 19. It is found that using linear PML as the absorbing boundary condition in nonlinear problems will cause a larger reflection error, more than one order of magnitude higher, than that using nonlinear PML.

III.D.2. Axisymmetric coordinate system

In this example, acoustic radiation from an oscillating circular piston in a wall is simulated. The axisymmetric nonlinear Euler equations and PML equations in primitive variables in the cylindrical coordinate system are
solved in the interior domain and the PML domain, respectively. Details on the derivation of the absorbing boundary condition are given in the Appendix. Velocity of the piston \( u = A \sin(\pi t/20) \), where \( A \) is a constant. The entire computational domain is \( 0 \leq z \leq 105, 0 \leq r \leq 105 \) with \( \Delta z = \Delta r = 0.5 \). A width of 10 grid points is applied in the PML domain. The wall and the piston are at \( z = 0 \), the cylindrical coordinate system is centered at the center of the piston. The radius of piston equals to 10. At the axis of symmetry \( r = 0 \), \( v \rightarrow 0 \), \( \frac{\partial v}{\partial r} \) are treated by the L'Hospital's rule, as \( r \rightarrow 0 \), \( v \rightarrow 0 \), \( \frac{\partial v}{\partial r} \rightarrow \frac{\partial v}{\partial r} \). If the amplitude of the piston velocity \( A \) is very small, it can be taken as a linearized problem, and the pressure distribution of acoustic radiation can be compared with the analytical solution. Figure 20 shows the fluctuation pressure distribution along the axis of the piston (\( r = 0 \)) when \( A = 10^{-4} \). Dashed lines indicate analytical solutions which are calculated by an integral formula.\(^{12}\) Good agreement is found.

As the amplitude of the piston oscillation increases, the nonlinearity of the waves increases. Figure 21 shows the pressure contours for \( A = 0.1 \) and 0.2. The agreement with the reference solution is very good. Figure 22 shows the reflection error as a function of time. Again, the error is small, indicating effectiveness of the boundary condition.

### IV. Conclusion

In many aeroacoustics problems, the nonlinear Euler equations are solved using the primitive variables. Although the Perfectly Matched Layer (PML) for the linearized Euler equations was first derived in primitive variables, its direct use in strongly nonlinear problems is not very effective, as demonstrated in this study. In this paper, PML absorbing boundary condition for use with nonlinear Euler equations in primitive variables has been derived. Two versions of PML equations for nonlinear Euler equations in primitive variables are given. One is in the two-dimensional Cartesian coordinate system, the other is for the axisymmetric nonlinear Euler equations. Although the nonlinear equations are not theoretically perfectly matched, the stability and effectiveness of the proposed PML as absorbing boundary condition are demonstrated by numerical examples, including absorption of an isentropic vortex, a nonlinear pressure pulse, roll-up vortices of a shear flow and acoustic radiation from an oscillating piston. Compared to its counterpart in conservation form, the primitive variable nonlinear PML absorbing boundary condition shows no substantial difference in performance. However, significant improvements over the linear PML equations have been observed. This shows the necessity of using nonlinear PML equations for nonlinear problems.

### Appendix

The axisymmetric nonlinear Euler equations written in primitive variables in the cylindrical coordinate system are

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial z} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial z} + \rho \frac{\partial v}{\partial r} + \rho \frac{\partial v}{\partial r} = 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0 \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial r} + \gamma p \frac{\partial u}{\partial r} + \frac{\partial \gamma p}{\partial r} = 0
\end{align*}
\]

where \( \rho \) is the density, \( u \) and \( v \) are velocity components in the axial \( z \) and radial \( r \) directions respectively, and \( p \) is the pressure.

We will consider the construction of PML equations in the \( z \) and \( r \) directions. Let

\[
\mathbf{u} = \begin{bmatrix}
\rho \\
u \\
v \\
p
\end{bmatrix}
= \begin{bmatrix}
\dot{\rho} \\
\dot{u} \\
\dot{v} \\
\dot{p}
\end{bmatrix}
+ \begin{bmatrix}
\rho' \\
u' \\
v' \\
p'
\end{bmatrix}
= \mathbf{\bar{u}} + \mathbf{u}'
\]

\( 12 \) of 28

American Institute of Aeronautics and Astronautics Paper 2009-0006
where a superscript “bar” indicates the mean flow and a superscript “prime” indicates the difference between the mean flow and the exact flow. The mean flow or pseudo mean flow should satisfy the steady Euler equations,

\[
\frac{\partial \bar{\rho}}{\partial t} + \bar{u} \frac{\partial \bar{\rho}}{\partial r} + \bar{v} \frac{\partial \bar{\rho}}{\partial z} + \rho \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{v}}{\partial z} = 0 \tag{60}
\]

\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial r} + \bar{v} \frac{\partial \bar{u}}{\partial z} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} = 0 \tag{61}
\]

\[
\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial r} + \bar{v} \frac{\partial \bar{v}}{\partial z} + \frac{1}{\rho} \frac{\partial \bar{u}}{\partial z} = 0 \tag{62}
\]

\[
\frac{\partial \bar{p}}{\partial t} + \bar{u} \frac{\partial \bar{p}}{\partial r} + \bar{v} \frac{\partial \bar{p}}{\partial z} + \gamma \bar{p} \frac{\partial \bar{u}}{\partial r} + \frac{\gamma \bar{p} \bar{v}}{\partial z} = 0 \tag{63}
\]

In an approach similar to that used in Section II, define new variables \(E, F\) and \(G\) as

\[
E = \frac{\partial \bar{u}}{\partial z} = \begin{bmatrix} \frac{\partial E}{\partial z} \\ \frac{\partial E}{\partial r} \\ \frac{\partial E}{\partial \theta} \end{bmatrix}, \quad F = \frac{\partial \bar{u}}{\partial r} = \begin{bmatrix} \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial \theta} \end{bmatrix}, \quad G = \begin{bmatrix} \frac{\partial G}{\partial z} \\ \frac{\partial G}{\partial r} \\ \frac{\partial G}{\partial \theta} \end{bmatrix} \tag{64}
\]

Then the Euler equations, (55)-(58), can be rewritten as

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + v \frac{\partial \rho}{\partial z} + \rho \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \rho \frac{\partial v}{\partial r} + G_{\rho} = 0 \tag{65}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} = 0 \tag{66}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial \rho}{\partial z} = 0 \tag{67}
\]

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + v \frac{\partial \rho}{\partial z} + \gamma \rho \frac{\partial u}{\partial r} + \frac{\gamma \rho \bar{v}}{\partial z} + G_{\rho} = 0 \tag{68}
\]

Subtracting out the mean flow, we have

\[
\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial r} + v \frac{\partial \rho'}{\partial z} + \rho \frac{\partial u'}{\partial r} + \frac{\partial v'}{\partial z} + \rho \frac{\partial v'}{\partial r} + G_{\rho} = 0 \tag{69}
\]

\[
\frac{\partial u'}{\partial t} + u \frac{\partial u'}{\partial r} + v \frac{\partial u'}{\partial z} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} = 0 \tag{70}
\]

\[
\frac{\partial v'}{\partial t} + u \frac{\partial v'}{\partial r} + v \frac{\partial v'}{\partial z} + \frac{1}{\rho} \frac{\partial \rho}{\partial z} = 0 \tag{71}
\]

\[
\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial r} + v \frac{\partial \rho'}{\partial z} + \gamma \rho \frac{\partial u'}{\partial r} + \frac{\gamma \rho \bar{v}}{\partial z} + G_{\rho} = 0 \tag{72}
\]

In cylindrical coordinates, the PML complex change of variables are

\[
z \rightarrow z + \frac{i}{\omega} \int \sigma_z dz, \quad \frac{\partial}{\partial z} \rightarrow \frac{1}{1 + \frac{i \sigma_z}{\omega}} \frac{\partial}{\partial z} \tag{73}
\]

\[
r \rightarrow r + \frac{i}{\omega} \int \sigma_r dr, \quad \frac{\partial}{\partial r} \rightarrow \frac{1}{1 + \frac{i \sigma_r}{\omega}} \frac{\partial}{\partial r} \tag{74}
\]

where \(\sigma_z\) and \(\sigma_r\) are absorption coefficients, which could be functions of \(z\) and \(r\), respectively, as shown in Figure 23.

After a process of derivation similar to those given in Section II, we get the following equations in the
frequency domain:

\[
\mathbf{E}' = \frac{1}{1 + \frac{i\sigma_p}{\omega}} \mathbf{\partial}(\mathbf{u} - \mathbf{u}) + \beta(-i\omega)(\mathbf{u} - \mathbf{u})
\] (75)

\[
\mathbf{F}' = \frac{1}{1 + \frac{i\sigma_p}{\omega}} \mathbf{\partial}(\mathbf{u} - \mathbf{u})
\] (76)

\[
\mathbf{G}' = \frac{1}{1 + \frac{i\sigma_p}{\omega} \int \sigma_r dr} \left[ \begin{array}{cc} \rho v - \rho \bar{v} & 0 \\ 0 & \gamma (\rho v - \rho \bar{v}) \end{array} \right]^T
\] (77)

where the superscript \( "T" \) denotes the transposition of the matrix.

In the unsplit approach, multiplying \( 1 + \frac{i\sigma_p}{\omega} \), \( 1 + \frac{i\sigma_p}{\omega} \) and \( 1 + \frac{i\sigma_p}{\omega} \) to (75), (76) and (77), respectively, we have

\[
\mathbf{E}' + \frac{i\sigma_p}{\omega} \mathbf{E}' = \frac{\partial (\mathbf{u} - \mathbf{u})}{\partial t} + \beta(-i\omega)(\mathbf{u} - \mathbf{u}) + \beta \sigma_p (\mathbf{u} - \mathbf{u})
\] (78)

\[
\mathbf{F}' + \frac{i\sigma_p}{\omega} \mathbf{F}' = \frac{\partial (\mathbf{u} - \mathbf{u})}{\partial t}
\] (79)

\[
\mathbf{G}' + \frac{i}{\omega} \int \sigma_r dr \mathbf{G}' = \left[ \begin{array}{cc} \rho v - \rho \bar{v} & 0 \\ 0 & \gamma (\rho v - \rho \bar{v}) \end{array} \right]^T
\] (80)

Now to rewrite (78)-(80) in the time domain, define

\[
\frac{\partial q_1}{\partial t} = E', \quad \frac{\partial q_2}{\partial t} = F', \quad \frac{\partial q_3}{\partial t} = G'
\] (81)

where \( q_1, q_2 \) and \( q_3 \) are auxiliary variables. The equations for auxiliary variables \( q_1, q_2 \) and \( q_3 \) are

\[
\frac{\partial q_1}{\partial t} + \sigma_z q_1 = \frac{\partial (\mathbf{u} - \mathbf{u})}{\partial t} + \beta \sigma_z (\mathbf{u} - \mathbf{u})
\] (82)

\[
\frac{\partial q_2}{\partial t} + \sigma_y q_2 = \frac{\partial (\mathbf{u} - \mathbf{u})}{\partial t}
\] (83)

\[
\frac{\partial q_3}{\partial t} + \frac{1}{r} \int \sigma_r dr q_3 = \left[ \begin{array}{cc} \rho v - \rho \bar{v} & 0 \\ 0 & \gamma (\rho v - \rho \bar{v}) \end{array} \right]^T
\] (84)

Finally, we have the PML equations in primitive variables as follows

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial z} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} - \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{v}}{\partial r} - \frac{\partial \bar{v}}{\partial z} + \frac{\partial v}{\partial r} + \frac{\partial v}{\partial z} = 0
\] (85)

\[
- \sigma_z \left[ uq_{1p} + \rho q_{1u} \right] + \beta \sigma_z \left[ u(p - \bar{p}) + \rho(u - \bar{u}) \right] - \sigma_r \left[ vq_{2p} + \rho q_{2u} \right] - \frac{q_3 p}{r} \int \sigma_r dr = 0
\] (86)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{u}}{\partial r} - \frac{1}{\rho} \frac{\partial \bar{u}}{\partial z} + \frac{1}{\rho} \frac{\partial \bar{u}}{\partial r} - \frac{1}{\rho} \frac{\partial \bar{u}}{\partial z} = 0
\] (87)

\[
- \sigma_z \left[ uq_{1v} + \beta \sigma_z u(v - \bar{v}) \right] - \sigma_r \left[ vq_{2v} + \frac{1}{\rho} q_{2p} \right] = 0
\] (88)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial r} + \frac{\gamma p}{\partial z} + \frac{\gamma p}{\partial r} - \frac{\partial u}{\partial z} - \frac{\partial \bar{u}}{\partial r} - \frac{\partial \bar{v}}{\partial z} - \frac{\partial \bar{v}}{\partial r} + \frac{\gamma pv}{\partial r} = 0
\] (89)

\[
- \sigma_z \left[ uq_{1p} + \gamma p q_{1u} \right] + \beta \sigma_z \left[ u(p - \bar{p}) + \gamma p(u - \bar{u}) \right] - \sigma_r \left[ vq_{2p} + \gamma p q_{2u} \right] - \frac{q_3 p}{r} \int \sigma_r dr = 0
\] (90)
Equations (85)-(88), (82)-(84) are solved in the PML domain.

The axisymmetric nonlinear Euler equations in primitive variables can also be rewritten in matrix form as

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial z} + B \frac{\partial u}{\partial r} + C \frac{u}{r} = 0 \tag{89}
\]

where

\[
C = \begin{bmatrix}
    v & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & v\gamma
\end{bmatrix}
\]

Then the axisymmetric nonlinear PML equations in primitive variables are as follows,

\[
\frac{\partial u}{\partial t} + A \left[ \frac{\partial u}{\partial z} + \beta \sigma_z (u - \bar{u}) - \sigma_z q_1 \right] + B \left[ \frac{\partial u}{\partial r} - \sigma_r q_2 \right] + C \frac{u}{r} - A \frac{\partial \bar{u}}{\partial z} - B \frac{\partial \bar{u}}{\partial r} - C \frac{\bar{u}}{r} = \int r \sigma_r dr q_3 \tag{90}
\]

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Figure 7. Pressure contour levels in every 0.01 from 0.66 to 1.40. \( P'_{\text{max}} = 1.0P_0 \). PML width \( D = 10\Delta x \).
Figure 8. Pressure along $y = 0$ at progressive frames. $P_{\text{max}}' = 1.0P_0$. PML width $D = 10\Delta x$. Solid line: numerical solution; dashed line: reference solution. Vertical dashed lines indicate the Euler/PML interface.
Figure 9. Maximum reflection error for the pressure as a function of time. $P'_{\text{max}} = 1.0P_0$. PML width $D$ is as indicated.

Figure 10. Maximum relative reflection error (pressure component) along lines $x = \pm 0.9$ and $y = \pm 0.9$ near the PML domain. PML width $D = 20\Delta x$. 
Figure 11. Maximum reflection error for the pressure as a function of pressure strength $P_{\text{max}}/P_0$. PML width $D = 20\Delta x$

Solid line with squares: in primitive variables; dashed line with circles: in conservation form.
Figure 12. Vorticity contours in progressive time frames.
Figure 13. Pressure and $v$-velocity at point $(x, y) = (8.75, 0)$ close to the outflow boundary. Solid line: computational; circle: larger domain calculation.
Figure 14. A schematic of acoustic radiation from an oscillating piston in a wall
Figure 15. Pressure contour levels in every 0.02 from 0.5 to 1.0, at the beginning of a cycle. $A = 0.2$. Solid line: computational; dashed line: large domain calculation.
Figure 16. Pressure along line $x = 0$ over a period of piston oscillation, from top to bottom. $A = 0.2, U_0 = 0$. Solid line: computational; dashed line: large domain calculation.
Figure 17. Maximum reflection error (pressure component) along lines $x = \pm 97.5$ and $y = 97.5$ close to the PML domain.
Figure 18. Pressure contour levels in every 0.02 from 0.5 to 1.0, at the beginning of a cycle, linear PML applied. \( A = 0.2, U_0 = 0.2 \). Solid line: computational; dashed line: large domain calculation.

Figure 19. Maximum reflection error (pressure component) along lines \( x = \pm 97.5 \) and \( y = 97.5 \) close to the PML domain. \( U_0 = 0.2 \).
Figure 20. Fluctuation pressure ($p - p_0$) distribution along the axis of the piston ($r = 0$), over a period of piston oscillation, from top to bottom. $A = 1.0 \times 10^{-4}$. Solid line: numerical solution; dashed line: analytical solution.

Figure 21. Pressure contours in every 0.01 from 0.6 to 9.0, at the beginning of a cycle. Solid line: computational; dashed line: large domain calculation.

(a) $A = 0.1$  
(b) $A = 0.2$
Figure 22. Maximum reflection error (pressure component) along lines $z = 97.5$ and $r = 97.5$ close to the PML domain.

Figure 23. PML in the cylindrical coordinate system. $z$—axial direction, $r$—radial direction.

(a) Schematics of physical and PML domains

(b) Absorption coefficients in the computational domain