Extremum Seeking for Model Reference Adaptive Control

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Adaptive control with predictable parameter convergence has remained a challenge for several decades. In the special case of set point adaptation, extremum seeking permits predictable parameter convergence by design. This is because persistency of excitation requirements are met by the sinusoidal perturbation that is part of the basic control design. Here we present results of extremum seeking based adaptation of model reference control of a simple roll rate model of a fixed wing aircraft. We show simulation results where adaptive tracking is achieved, and the convergence of parameters conforms to predictions from the theory of extremum seeking. In the case of actuator failure, we show that the parameter convergence proofs of extremum seeking are directly applicable.

I. INTRODUCTION

Adaptive control schemes2,4,6 provide exponential stability of the homogenous error system under conditions of persistency of excitation. But the size of the exponents depends upon the initial conditions of parameter estimation error, and hence predictable performance is never available. While extremum seeking1 provides predictable performance, it only adapts the set point of a control system and proofs of extremum seeking performance are not available for settings where plant parameters other than the unknown set point vary.

In this work, we adapt a Model Reference Control law using extremum seeking—we will refer to this scheme as ES-MRAC henceforth (Extremum Seeking-Model Reference Adaptive Control). We supply simulations that show parameter convergence of the adaptation conforming to predictions from the theory of extremum seeking. In the case of actuator failure, we show that we can design the adaptation transient within a priori bounds. The results in this paper thus point to the possibility of controlling the convergence rates of the parameters in adaptive control with a time varying adaptive controller.

The paper is organized as follows. Section II sums up results from the theory of single parameter extremum seeking and corresponding design guidelines from Ariyur and Krstic.† Section III presents adaptation of model reference control via extremum seeking, Section IV supplies simulation results of the scheme, Section V supplies stability analysis of the scheme, and Section VI provides the case where there is only an actuator failure, and the results of extremum seeking theory are directly applicable.

II. EXTREMUM SEEKING CONTROL

The mainstream methods of adaptive control for linear2,6 and nonlinear4 systems are applicable only for regulation to known set points or reference trajectories. In some applications, the reference-to-output map has an extremum (i.e., a maximum or a minimum) and the objective is to select the set point to keep
Figure 1. Extension of the extremum seeking algorithm to non-step changes in $\theta^*$ and $f^*$

the output at the extremum value. The uncertainty in the reference-to-output map makes it necessary to use some sort of adaptation to find the set point which extremizes (maximizes/minimizes) the output. The emergence of extremum control dates as far back as the 1922 paper of Leblanc,\(^5\) whose scheme may very well have been the first “adaptive” controller reported in the literature. The method of sinusoidal perturbation used in this work has been the most popular of extremum-seeking schemes. In fact, it is the only method that permits fast adaptation, going beyond numerically based methods that need the plant dynamics to settle down before optimization. In this section, we provide the background and sum up the design results for extremum seeking control.\(^1\) Figure 1 shows the nonlinear plant with linear dynamics along with the extremum seeking loop. We let $f(\theta)$ be a function of the form:

$$ f(\theta) = f^*(t) + \frac{f''}{2} (\theta - \theta^*(t))^2, $$

where $f'' > 0$ is constant but unknown. The purpose of extremum seeking is to make $\theta - \theta^*(t)$ as small as possible, so that the output $F_o(s)[f(\theta)]$ is driven to its extremum $F_o(s)[f^*(t)]$. The optimal input and output, $\theta^*$ and $f^*$, are allowed to be time varying. Let us denote their Laplace transforms by

$$ \mathcal{L}\{\theta^*(t)\} = \lambda_\theta \Gamma_\theta(s) $$
$$ \mathcal{L}\{f^*(t)\} = \lambda_f \Gamma_f(s). $$

If $\theta^*$ and $f^*$ happen to be constant (step functions),

$$ \mathcal{L}\{\theta^*(t)\} = \frac{\lambda_\theta}{s} $$
$$ \mathcal{L}\{f^*(t)\} = \frac{\lambda_f}{s}. $$

While $\lambda_\theta$ and $\lambda_f$ are unknown, the Laplace transform (qualitative) form of $\theta^*$ and $f^*$ is known, and is included in the washout filter

$$ C_o(s) \Gamma_f(s) = \frac{s}{s + h} $$

(where in the static case we had chosen $C_o(s) = 1/(s + h)$) and in the estimation algorithm

$$ C_i(s) \Gamma_\theta(s) = \frac{1}{s} $$

(where in the static case we had chosen $C_i(s) = 1$). Let us first shed more light on $\Gamma_\theta(s)$ and $\Gamma_f(s)$ and then return to discuss $C_i(s)$ and $C_o(s)$.

By allowing $\theta^*(t)$ and $f^*(t)$ to be time varying, we are allowing for the possibility of having to optimize a system whose commanded operation is non-constant. For example, if we have to ramp up the power of a gas turbine engine, we know the shape of, say, $f^*(t)$,

$$ \mathcal{L}\{f^*(t)\} = \frac{\lambda_f}{s^2}. $$
but we don’t know \( \lambda_f \) (and \( \lambda_0 \)). We include \( \Gamma_f(s) = 1/s^2 \) into the extremum seeking scheme to compensate for the fact that \( f^* \) is not constant. The inclusion of \( \Gamma_f(s) \) and \( \Gamma_\theta(s) \) into the respective blocks in the feedback branch of Figure 1 follows the well known internal model principle. In its simplest form, this principle guides the use of an integrator in a PI controller to achieve a zero steady-state error. When applied, in a very generalized manner, to the extremum seeking problem, it allows us to track time-varying maxima or minima.

We return now to the compensators \( C_o(s) \) and \( C_i(s) \). Their presence is motivated by the dynamics \( F_o(s) \) and \( F_i(s) \), but also by the reference signals \( \Gamma_\theta(s) \) and \( \Gamma_f(s) \). For example, if we are tracking an input ramp,

\[
\Gamma_\theta(s) = \frac{1}{s^2},
\]

we get a double integrator in the feedback loop, which poses a threat to stability. Rather than choosing \( C_i(s) = 1 \), we would choose \( C_i(s) = s + 1 \) (or something similar) to reduce the relative degree of the loop. The compensators \( C_i(s) \) and \( C_o(s) \) are crucial design tools for satisfying stability conditions and achieving desired convergence rates.

We now make assumptions upon the system in Figure 1 that underlie the analysis leading to the design theorem:

**Assumption II.1** \( F_i(s) \) and \( F_o(s) \) are asymptotically stable and proper.

**Assumption II.2** \( \Gamma_f(s) \) and \( \Gamma_\theta(s) \) are strictly proper rational functions and poles of \( \Gamma_\theta(s) \) that are not asymptotically stable are not zeros of \( F_i(s) \).

This assumption forbids delta function variations in the map parameters and also the situation where tracking of the extremum is not possible.

**Assumption II.3** \( \frac{C_o(s)}{\Gamma_f(s)} \) and \( C_i(s)\Gamma_\theta(s) \) are proper.

This assumption ensures that the filters \( \frac{C_o(s)}{\Gamma_f(s)} \) and \( C_i(s)\Gamma_\theta(s) \) in Figure 1 can be implemented. Since \( C_i(s) \) and \( C_o(s) \) are at our disposal to design, we can always satisfy this assumption. The analysis does not explicitly place conditions upon the dynamics of the parameters \( \Gamma_\theta(s) \) and \( \Gamma_f(s) \), however, for any design to yield a nontrivial region of attraction around the extremum, they cannot be faster than plant dynamics in \( F_i(s) \) and \( F_o(s) \). The signal \( n \) in Figure 1 denotes the measurement noise.

### A. Single Parameter Stability Test

We first provide background for the result on output extremization below. The equations describing the single parameter extremum seeking scheme in Fig. 1 are:

\[
y = F_o(s) \left[ f^*(t) + \frac{f''}{2} (\theta - \theta^*(t))^2 \right] \quad (2)\\
\theta = F_i(s) \left[ a \sin(\omega t) - C_i(s)\Gamma_\theta(s)[\xi] \right] \quad (3)\\
\xi = k \sin(\omega t - \phi) \frac{C_o(s)}{\Gamma_f(s)}[y + n]. \quad (4)
\]

For the purpose of analysis, we define the tracking error \( \hat{\theta} \) and output error \( \hat{y} \):

\[
\hat{\theta} = \theta^*(t) - \theta + \theta_0 \quad (5)\\
\theta_0 = F_i(s) [a \sin(\omega t)] \quad (6)\\
\hat{y} = y - F_o(s) [f^*(t)]. \quad (7)
\]

In terms of these definitions, we can restate the goal of extremum seeking as driving output error \( \hat{y} \) to a small value by tracking \( \theta^*(t) \) with \( \theta \). With the present method, we cannot drive \( \hat{y} \) to zero because of the sinusoidal perturbation \( \theta_0 \).
We provide a result below that permits systematic design in a variety of situations. To this end, we introduce the following notation:

\[
H_i(s) = C_i(s)\Gamma_\theta(s)F_i(s)
\]

(8)

\[
H_o(s) = k\frac{C_o(s)}{\Gamma_f(s)}F_o(s)
\]

(9)

\[
H_o(s) = k\frac{C_o(s)}{\Gamma_f(s)}F_o(s) \triangleq H_{osp}(s)H_{obp}(s) \triangleq H_{osp}(s)(1 + H_{obp}^*(s))
\]

(10)

where \(H_{osp}(s)\) denotes the strictly proper part of \(H_o(s)\) and \(H_{obp}(s)\) its biproper part, and \(k\) is chosen to ensure

\[
\lim_{s \to 0} H_{osp}(s) = 1.
\]

(11)

Now we make an additional assumption upon the plant:

**Assumption II.4** Let the smallest in absolute value among the real parts of all of the poles of \(H_{osp}(s)\) be denoted by \(a\). Let the largest among the moduli of all of the poles of \(F_i(s)\) and \(H_{obp}(s)\) be denoted by \(b\). The ratio \(M = a/b\) is sufficiently large.

The purpose of this assumption is to use a singular perturbation reduction of the output dynamics and provide the LTI SISO stability test of the theorem stated below. If the assumption were made upon the output dynamics \(F_o(s)\) alone, the design would be restricted to plants with fast output dynamics \(F_o(s)\). Hence, for generality in the design procedure, the assumption of fast poles is made upon the strictly proper factor \(H_{osp}(s)\) of \(H_o(s)\). Its purpose is to deal with the strictly proper part of \(F_o(s)\). If we have slow poles in a strictly proper \(F_o(s)\), we can introduce a biproper \(\frac{C_o(s)}{\Gamma_f(s)}\) with an equal number of fast poles to permit analysis based design. For example, if

\[
F_i(s) = \frac{1}{s + 1}, \quad \text{and} \quad F_o(s) = \frac{1}{(s + 1)(2s + 3)},
\]

with constant \(f^*\) and \(\theta^*\) (giving \(\Gamma_\theta(s) = \Gamma_f(s) = 1/s\)) we may set

\[
C_o(s) = \frac{(s + 4)}{(s + 5)(s + 6)}
\]

and \(k = 60\) to give

\[
H_o(s) = \frac{C_o(s)}{\Gamma_f(s)}F_o(s) = \frac{60s(s + 4)}{(s + 1)(2s + 3)(s + 5)(s + 6)}.
\]

We can factor the fast dynamics as

\[
H_{osp}(s) = \frac{30}{(s + 5)(s + 6)}
\]

and the slow biproper dynamics as

\[
H_{obp}(s) = 1 + H_{obp}^*(s) = 1 + \frac{1.5(s - 1)}{(s + 1)(s + 1.5)}.
\]

This gives, in the terms of Assumption II.4, the smallest pole in absolute value in \(H_{osp}(s)\), \(a = 5\), the largest of the moduli of poles in \(F_i(s)\) and \(H_{obp}(s)\) as \(b = 1.5\), giving their ratio \(M = a/b = 3.33\). The singular perturbation reduction reduces the fast dynamics \(H_{osp}(s) = \frac{30}{(s+5)(s+6)}\) to its unity gain, and we deal with stability of the reduced order model via the method of averaging to deduce stability conditions for the overall system in the theorem below.

**Theorem II.1 (Single Parameter Extremum Seeking)** For the system in Figure 1, under Assumptions II.1–II.4, the output error \(\hat{y}\) achieves local exponential convergence to an \(O(a^2 + \delta^2)\) neighborhood of the origin, where \(\delta = 1/\omega + 1/M\) provided \(n = 0\) and:

1. Perturbation frequency \(\omega\) is sufficiently large, and \(\pm j\omega\) is not a zero of \(F_i(s)\).
2. Zeros of $\Gamma_1(s)$ that are not asymptotically stable are also zeros of $C_o(s)$.

3. Poles of $\Gamma_0(s)$ that are not asymptotically stable are not zeros of $C_i(s)$.

4. $C_o(s)$ and $\frac{1}{\Gamma_o(s)}$ are asymptotically stable, where

$$L(s) = \frac{af''}{4}\text{Re}\{e^{j\phi}F_i(j\omega)\}H_i(s).$$

(12)

From Eqn. (12), we notice that $C_i(s)$ appears linearly in $L(s)$ (through $H_i(s) = C_i(s)\Gamma_0(s)F_i(s)$). This property allows systematic design using linear control tools. The conditions of Theorem II.1 motivate the steps of a design algorithm below.

B. Single Parameter Compensator Design

In the design guidelines that follow, we set $\phi = 0$ which can be used separately for fine-tuning.

Algorithm II.1 (Single Parameter Extremum Seeking)

1. Select the perturbation frequency $\omega$ sufficiently large and not equal to any frequency in noise, and with $\pm j\omega$ not equal to any imaginary axis zero of $F_i(s)$.

2. Set perturbation amplitude $a$ so as to obtain small steady state output error $\tilde{y}$.

3. Design $C_o(s)$ asymptotically stable, with zeros of $\Gamma_1(s)$ that are not asymptotically stable as its zeros, and such that $\frac{C_o(s)}{\Gamma_1(s)}$ is proper. In the case where dynamics in $F_o(s)$ are slow and strictly proper, use as many fast poles in $C_o(s)$ as the relative degree of $F_o(s)$, and as many zeros as needed to have zero relative degree of the slow part $H_{obs}(s)$ to satisfy Assumption II.4.

4. Design $C_i(s)$ by any linear SISO design technique such that it does not include poles of $\Gamma_0(s)$ that are not asymptotically stable as its zeros, $C_i(s)\Gamma_0(s)$ is proper, and $\frac{1}{\Gamma_i(s)}$ is asymptotically stable.

We examine these design steps in detail:

Step 1: Since the averaging assumption is only qualitative, we may be able to choose $\omega$ only slightly larger than the plant time constants. Choice of $\omega$ equal to a frequency component persistent in the noise $n$ can lead to a large steady state tracking error $\tilde{\theta}$. In fact, Theorem II.1 can be adapted to include a bounded noise signal satisfying $\lim_{t \to \infty} \frac{1}{T} \int_0^T n \sin \omega dt = 0$. Finally, if $\pm j\omega$ is a zero of $F_i(s)$, the sinusoidal forcing will have no effect on the plant.

Step 2: The perturbation amplitude $a$ should be designed such that $a|F_i(j\omega)|$ is small; $a$ itself may have to be large so as to produce a measurable variation in the plant output.

Step 3: In general, this design step will need designing a biproper $\frac{C_o(s)}{\Gamma_1(s)}$ when we have a slow and strictly proper $F_o(s)$ in order to satisfy Assumption II.4. The use of fast poles in $\frac{C_o(s)}{\Gamma_1(s)}$ raises a possibility of noise deteriorating the feedback; however, the demodulation coupled with the integrating action of the input compensator prevents frequencies other than that of the forcing from entering into the feedback. While we have used the gain $k$ in analysis to satisfy Assumption II.4, this is not strictly necessary in design.

Step 4: We see from Algorithm II.1 that $C_i(s)$ has to be designed such that $C_i(s)\Gamma_0(s)$ is proper; hence, for example, if $\Gamma_0(s) = \frac{1}{s}$, an improper $C_i(s) = 1 + d_1s + d_2s^2$ is permissible. In the interest of robustness, it is desirable to design $C_i(s)$ and $C_o(s)$ to ensure minimum relative degree of $C_i(s)\Gamma_0(s)$ and $\frac{C_o(s)}{\Gamma_1(s)}$. This will help to provide lower loop phase and thus better phase margins. Simplification of the design for $C_i(s)$ is achieved by setting $\phi = -\angle(F_i(j\omega))$, and obtaining

$$L(s) = \frac{af''|F_i(j\omega)|}{4}H_i(s).$$

The attraction of extremum seeking is its ability to deal with uncertain plants. In our design, we can accommodate uncertainties in $f''$, $F_o(s)$, and $F_i(s)$, which appear as uncertainties in $L(s)$. Methods for treatment of these uncertainties are dealt with in texts such as. Here we only show how it is possible to ensure robustness to variations in $f''$. Let $f''$ denote an a priori estimate of $f''$. Then we can represent
\[
\frac{1}{1+L(s)} \frac{1}{1+L(s)} = \frac{1}{1+\left(1+\frac{\Delta f″}{f″}\right)P(s)}
\]
where \(\Delta f″ = f″ - \hat{f}″\), and \(P(s) = \frac{P}{1+P}L(s)\), which is at our disposal because \(f″\) in \(P(s)\) gets cancelled by \(f″\) in \(L(s)\). We design \(C_i(s)\) to minimize \(\|P_1+P\|_{H_\infty}\) which maximizes the allowable \(\Delta f″ < \hat{f}″ / \|P_1+P\|_{H_\infty}\) under which the system is still asymptotically stable.

### III. ES-MRAC: ADAPTING MODEL REFERENCE CONTROL VIA EXTREMUM SEEKING

Figure 2 shows the ES-MRAC scheme for the roll rate model with extremum seeking for both parameters. It uses the reference model,

\[
\dot{x}_m = a_m x_m + b_m r
\]

for the roll rate model,

\[
\dot{x} = ax + bu
\]

with control input

\[
u = k_x x + k_r r
\]

where \(r\) is the reference setting, and \(x\) is the roll rate. The model reference error is defined as

\[
e \triangleq x - x_m
\]

If we define the ideal coefficients \(k_x^*\) and \(k_r^*\) as the ones that get the system to match the reference model, we have

\[
k_x^* = (a_m - a) / b, \quad k_r^* = b_m / b
\]

We now consider the application of extremum seeking to optimize the value of a suitable function of the error \(e\), which we will denote by \(y = f(e) = e^2 / 2\). In this problem, the optimum \(f^* = 0\), and \(y\) is subject to step changes if we assume that the reference input \(r\) is a step. The extremum seeking scheme has the standard configuration of a washout filter, modulation and demodulation and an integrator for parameter tracking for each parameter in the control law, with a compensator to provide damping (\(d_1\) and \(d_2\)).

### IV. ESMRAC DESIGN AND SIMULATION RESULTS

Figures 3, 4, and 5 show the performance of the system of Figure 2 when the design parameters are chosen as follows according to the design algorithm II.1: perturbation amplitudes \(a_1 = a_2 = 0.3\); perturbation frequencies \(\omega_1 = 8\text{rad/sec}, \omega_2 = 11\text{rad/sec}\); gains \(k_x = 2000, k_r = 4000\); damping \(d_1 = d_2 = 0.1\); and washout filter poles \(h_1 = 4\text{rad/sec}, h_2 = 5.6\text{rad/sec}\). The stability theorem for single parameter extremum seeking II.1 yields an approximate estimate of the exponent of the closed loop system at around -0.7283 (taking into account the true plant and an actuator pole assumed at -20 rad/sec and substituting into Eqn. (12)) in steady state (assume we start with the true plant close to the reference plant). This in turn yields a settling time of around 4 seconds, a fact borne out in the simulations.

### V. ANALYSIS OF THE ES-MRAC SCHEME

The object of the ES-MRAC scheme in Figure 2 is to ensure zero model reference error and adapt the control gains \(k_x\) and \(k_r\) in the control law

\[
u = k_x x + k_r r
\]

to their ideal values

\[
k_x^* = \frac{a_m - a}{b}, \quad k_r^* = \frac{b_m}{b}
\]

so that the output of the plant

\[
\dot{x} = ax + bu
\]
Figure 2. ES-MRAC Scheme
matches the output of the reference model

\[ \dot{x}_m = a_m x_m + b_m r \]

for the reference input \( r \).

Extremum seeking does not seek to obtain exact convergence of the parameters to their ideal values, but rather to obtain convergence of their average values to the ideal values. As can be seen from the figure, the scheme involves continuously perturbing the parameter values in order to change them to minimize the cost function \( y = f(e) \). We considered several cost functions in our simulation studies:

\[ y = e^2, \quad y = \dot{e}^2, \quad y = |e| \]

All of these produced consistent convergence of adaptation. Given this success, there arises the question whether rigorous stability proofs exist for these schemes under a variety of conditions, which in turn would permit systematic design of the adaptation parameters in the extremum seeking—\( \omega_1, a_1, \phi_1, h_1, \) and \( g_1 \) for the \( k_x \) adaptation, and \( \omega_2, a_2, \phi_2, h_2, \) and \( g_2 \) for the \( k_r \) adaptation. We perform analysis of the scheme along the lines of analysis performed for extremum seeking schemes. For analysis of the ES-MRAC scheme
in Figure 2, we write down its governing equations as follows:

\[ \dot{k}_x = -\frac{g_1}{s} \xi_1 a_1 \sin(\omega_1 t) \sin(\omega_1 t - \phi_1) \frac{s}{s + h_1} \frac{b_m}{s - a_m} \]  

(17)

\[ \dot{k}_r = -\frac{g_2}{s} \xi_2 a_2 \sin(\omega_2 t) \sin(\omega_2 t - \phi_2) \frac{s}{s + h_2} \]  

(18)

\[ \dot{x} = ax + bu \]  

(19)

\[ y = f(e, \dot{e}) \]  

(20)

\[ \dot{k}_x = -\frac{g_1}{s} \xi_1 a_1 \sin(\omega_1 t) \sin(\omega_1 t - \phi_1) \frac{s}{s + h_1} \frac{b_m}{s - a_m} \]  

(21)

\[ \dot{k}_r = -\frac{g_2}{s} \xi_2 a_2 \sin(\omega_2 t) \sin(\omega_2 t - \phi_2) \frac{s}{s + h_2} \]  

(22)

To examine convergence of the extremum seeking loops as in Section II, we define error variables \( \tilde{k}_x \) and \( \tilde{k}_r \), perturbation signals \( k_{x0} \) and \( k_{r0} \), and tracking error \( \tilde{y} \):

\[ \tilde{k}_x \equiv k^*_x - k_x + k_{x0} \]  

\[ \tilde{k}_r \equiv k^*_r - k_r + k_{r0} \]  

\[ k_{x0} \equiv a_1 \sin(\omega_1 t), \quad k_{r0} \equiv a_2 \sin(\omega_2 t) \]  

\[ \tilde{y} \equiv y - f^* = y \quad (\text{assuming} \quad f^* = 0), \]  

(23)

The equation for the model reference error \( e \) can then be written as:
\[
\dot{e} = (a_m + b \left( k_{x0} - \tilde{k}_x \right)) e + b \left( k_{x0} - \tilde{k}_x \right) x_m + b \left( k_{r0} - \tilde{k}_r \right) r,
\]

The reference trajectory \(x_m\) is governed by the reference model:

\[
\dot{x}_m = a_m x_m + b_m u
\]

And the equations for the gains can be written as:

\[
\begin{align*}
 k_x &= k_{x0} - \frac{q_1}{s} \left[ \sin (\omega_1 t - \phi_1) \xi_1 \right], \\
 k_r &= k_{r0} - \frac{q_2}{s} \left[ \sin (\omega_2 t - \phi_2) \xi_2 \right]
\end{align*}
\]

\[
\begin{align*}
 \xi_1 &= s \frac{h_1}{s + h_1} \left[ y \right], \\
 \xi_2 &= s \frac{h_2}{s + h_2} \left[ y \right]
\end{align*}
\]

We write the outputs of the washout filters in the following form for analysis:

\[
\begin{align*}
 \xi_1 &= \left( 1 - \frac{h_1}{s + h_1} \right) \left[ y \right] = y - \xi_1, \\
 \xi_2 &= \left( 1 - \frac{h_2}{s + h_2} \right) \left[ y \right] = y - \xi_2
\end{align*}
\]

We can now write the equations for the parameter tracking error variables as follows:

\[
\begin{align*}
 \dot{\tilde{k}}_x &= g_1 \left[ \sin (\omega_1 t - \phi_1) \left( y - \xi_1 \right) \right] \\
 \dot{\tilde{k}}_r &= g_2 \left[ \sin (\omega_2 t - \phi_2) \left( y - \xi_2 \right) \right]
\end{align*}
\]

Figure 5. Parameter convergence
A. AVERAGING ANALYSIS

As with the analysis of extremum seeking, we apply the method of averaging (see for example, Chapter 9 in the book Nonlinear Systems, by H. K. Khalil) to study the stability of the ES-MRAC scheme. The terms $\xi_1$ and $\xi_2$ fall out as the terms containing them average out to zero in the parameter tracking equations. So we only need consider four equations, and given that the reference model is independent of the other three, we have only three differential equations.

The period of averaging is taken as the lowest common multiple of the periods of the two perturbation frequencies $\omega_1$ and $\omega_2$. If the time period of averaging, we have $T = p\frac{2\pi}{\omega_1} = q\frac{2\pi}{\omega_2}$, where $p$ and $q$ are natural numbers. Before performing the averaging, we perform a scaling of the time unit to obtain the small parameter used for averaging. We set $\tau = \omega_1 t$ where we assume that $\omega_1 < \omega_2$ and obtain a transformed set of governing equations:

\[
\begin{align*}
\frac{de}{dT} &= \frac{1}{\omega_1} \left( a_m + b \left( a_1 \sin \tau - \tilde{k}_x \right) \right) e + b \left( a_1 \sin \tau - \tilde{k}_x \right) x_m + b \left( a_2 \sin \frac{2}{p} \tau - \tilde{k}_r \right) r \\
\frac{d\tilde{k}_x}{dT} &= \frac{1}{\omega_1} g_1 \sin (\tau - \phi_1) \left( y - \xi_1 \right) \\
\frac{d\tilde{k}_r}{dT} &= \frac{1}{\omega_1} g_2 \sin \left( \frac{2}{p} \tau - \phi_2 \right) \left( y - \xi_2 \right)
\end{align*}
\]

The above set of equations is time varying and nonlinear; we arrive at an averaged system of equations by integrating the right hand side of the equations over the period $T$, taking the state variables constant. In all of the cases, we need to use higher order averaging theorems to establish stability.

For the cost function $y = e^2$, we get the following averaged equations (after a long sequence of calculations):

\[
\begin{align*}
\frac{de_{av}}{dT} &= \frac{1}{\omega_1} \left[ \left( a_m - b \tilde{k}_x_{av} \right) e_{av} - b \tilde{k}_x_{av} x_{m,av} - b \tilde{k}_r_{av} r \right] \\
\frac{dk_{x,av}}{dT} &= \frac{1}{\omega_1} g_1 \cos \phi_1 \sin (\tau - \phi_1) \left( e_{av} + x_{m,av} \right) \left[ \left( a_m - b \tilde{k}_x_{av} \right) e_{av} - b \tilde{k}_x_{av} x_{m,av} - b \tilde{k}_r_{av} r \right] \\
\frac{dk_{r,av}}{dT} &= \frac{1}{\omega_1} g_2 \cos \phi_2 \sin \left( \frac{2}{p} \tau - \phi_2 \right) \left[ \left( a_m - b \tilde{k}_x_{av} \right) e_{av} - b \tilde{k}_x_{av} x_{m,av} - b \tilde{k}_r_{av} r \right] \\
\frac{dx_{m,av}}{dT} &= \frac{1}{\omega_1} \left[ a_m x_{m,av} + b_m r \right]
\end{align*}
\]

The averaged error equation is the same for all cost functions based on $e$, and for all of the cost functions considered, we found equilibria at

$$e_{av} = 0, \quad \tilde{k}_{r,av} = \frac{b_m}{a_m} \tilde{k}_{x,av},$$

for the error system using the equilibrium $x_{m,av} = -\frac{b_m}{a_m} r$ for the reference model. This might account for some of our simulation results, where the parameter estimates converge to values other than the expected ideal values, while the error still converges to zero. Nevertheless, convergence in simulation to the ideal parameter values for a variety of initial conditions suggests a stability proof exists for some set of initial conditions in parameter estimation errors.

VI. ACTUATOR FAILURE

In the case where $a_m = a$ in Figure 2, and there is only a change in $b$, e.g., a degradation in the actuator, the adaptation problem is almost identical to the extremum seeking problem of set point optimization. The only difference is the presence of an unknown gain in the loop (the value of $b$). The method at the end of Section II can be used to design a robust extremum seeking loop and we can know the exponent of parameter convergence to within an interval of uncertainty determined by the interval of uncertainty of $b$.

VII. CONCLUDING REMARKS

This work raises two key questions. The first is whether it is possible to achieve completely controlled parameter convergence in adaptive control with a time varying adaptive controller. The second is what is the structure of the measurements and control inputs that will permit this predictable convergence. We will complete this analysis to obtain conditions for global stability of parameter tracking. Indeed, we have these exponents from linearization of the averaged equations. For the case where only the change in $b$ or actuator failure needs adaptation to, the proofs of the standard extremum seeking techniques already hold.
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References


