Determination of Acoustic Transfer Matrices via Large Eddy Simulation and System Identification

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The transfer behavior of acoustic waves at a sudden change in cross section in a duct system is investigated. In particular, a method is presented which allows to determine the coefficients of transmission and reflection of plane acoustic waves by combining large eddy simulation (LES) of turbulent compressible flow with system identification. The complex aeroacoustic interactions between acoustic waves and free shear layers are captured in detail, such that the transfer coefficients can be determined accurately from first principles. The method works as follows: At first, an LES with external, broadband excitation of acoustic waves is carried out. Time series of acoustic data are extracted from the computed flow field and analyzed with system identification techniques in order to determine the acoustic transfer coefficients for a range of frequencies. The combination of the broadband excitation with highly parallelized LES makes the overall approach quite efficient, despite the difficulties associated with simulation of low-Mach number compressible flows. In order to demonstrate the reliability and accuracy of the method, the results for the transfer behavior and the acoustic impedance are presented and physically interpreted in combination with several analytical models and experimental data.

I. Introduction

In contemporary urban environments, noise emissions from, e.g., transportation systems or air conditioning equipment have become detrimental to the health and well-being of millions of people. Consequently, the field of aeroacoustic analysis and design, which is decisive for noise prediction and reduction, has become of increasing importance. The development of novel methods, which aim for an efficient design process that meets strict acoustical design objectives, has become a challenging focus of modern engineering sciences.

A common approach to study the propagation and dissipation of sound in duct systems, as used for the design of mufflers, after treatment devices in exhaust systems, or ventilation systems, utilizes low-order acoustic network models (e.g. Åbom and Boden1 or Munjal2). This approach allows for a quick assessment of the acoustic behavior under different operating conditions and for different geometries. The models consist of an assembly of discrete acoustic elements, i.e. “acoustic two-ports”, which are represented mathematically by an acoustic scattering or transfer matrix. Of course, in order to predict the overall system behavior correctly, the matrix coefficients of the individual acoustic elements have to be known. Very simple elements like ducts can be described analytically. However, the accurate representation of the geometries involving complex turbulent acoustic interactions requires advanced numerical methods. For this reason, a methodology has been developed to determine the acoustic transfer behavior of elements involving compressible turbulent wall-bounded flows. This approach combines acoustically excited, compressible large eddy simulations and system identification methods from signal processing in order to obtain accurate low-order element descriptions.

II. Background

In the following, a brief introduction to the acoustic element notation and the possible approaches for determining their acoustic transfer behavior are discussed.

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In linear duct acoustics, the transmission, reflection and absorption of plane, harmonic waves at frequencies below the first cut-off frequency are described completely by the so-called “scattering matrices”

\[
\begin{pmatrix}
  f_d \\
  g_u
\end{pmatrix} = 
\begin{pmatrix}
  t_u & r_d \\
  r_u & t_d
\end{pmatrix} 
\begin{pmatrix}
  f_u \\
  g_d
\end{pmatrix}.
\]  

(1)

The characteristic wave amplitudes \(f\) and \(g\) represent plane acoustic waves travelling along the duct in the downstream and upstream directions, respectively (see Fig. 2). They can be related to the acoustic velocity fluctuations \(u'\) and the acoustic pressure fluctuations \(p'\) by the following relations:

\[
f = \frac{1}{2} \left( \frac{p'}{\rho c} + u' \right),
\]

(2)

\[
g = \frac{1}{2} \left( \frac{p'}{\rho c} - u' \right),
\]

(3)

with \(\rho\) representing the density and \(c\) the speed of sound.

The coefficients of the scattering matrix (1) can be interpreted as transmission and reflection coefficients \(t\) and \(r\) for the characteristic wave amplitudes \(f\) and \(g\) in the down- and upstream direction (subscripts \(d\) and \(u\)), respectively. The coefficients depend strongly on the given geometry, flow velocities, speed of sound and on the frequency of the impinging waves (e.g. for a sudden area expansion\(^4,5\)).

Another common description of the acoustical transfer behavior of systems are the so-called “transfer matrices” (Eq. 4). The relation could be obtained when equations (2) and (3) are applied on the scattering matrix (Eq. 1):

\[
\begin{pmatrix}
  p' \\
  u'
\end{pmatrix}_d = 
\begin{pmatrix}
  T_{11} & T_{12} \\
  T_{21} & T_{22}
\end{pmatrix} 
\begin{pmatrix}
  p' \\
  u'
\end{pmatrix}_u.
\]

(4)

The coefficients of the scattering/transfer matrix can be determined by analytical, experimental or numerical methods. Since analytical methods\(^4,6,7\) are restricted to very simple configurations, like ducts or discrete area changes, only experimental approaches (e.g. the two-source method\(^1,8\)) or numerical, CFD-based techniques,\(^9-11\) are applicable to more complex geometries and flow fields.

Polifke and co-workers\(^9,10,12\) have proposed a hybrid CFD/SI approach, where time series data from a single CFD simulation are post-processed with system identification tools in order to determine transfer matrix coefficients over a broad frequency range. Until recently, applications of CFD/SI were based on Unsteady Reynolds-averaged Navier-Stokes (URANS) simulations,\(^10\) which present limitations to certain types of turbulent flows (e.g. flow separations, free shear layers, swirling flows with vortex breakdown or turbulent combustion). Hence, it is expected that the CFD/SI approach based on URANS cannot provide accurate enough results for the acoustic transmission behavior in complex turbulent flows, because of the interaction between acoustical waves and an erroneous mean flow field. Furthermore, direct interactions of acoustic waves and turbulent vortices are not captured by relying on a time and space averaged turbulence model. To overcome these limitations, a method that combines large eddy simulations (LES) and system identification methods (SI) has been developed. The LES approach resolves the large-scale, energy containing turbulent flow structures, whereas the small-scale turbulent fluctuations and their effects on the resolved large-scale vortices are represented by a Sub-Grid-Scale (SGS) model.

The LES/SI methodology has been applied to analyze the acoustic scattering behavior of elements in pipe systems, e.g. sudden area expansions,\(^13\) cylindrical orifices\(^14\) and T-junctions.\(^15\) Moreover, the use of the LES/SI methodology for the identification of flame transfer functions (of crucial importance for the analysis of thermo-acoustic instabilities) is discussed in Giauque et al.,\(^16\) Kaess et al.\(^17\) and Tay Wo Chong et al.\(^18\).

In this paper, the acoustic behavior of the cylindrical backward-facing step (Fig. 1) is further investigated utilizing the LES/SI approach. The quantitative accuracy of the results presented in a previous publication\(^13\) were not entirely satisfactory. The present paper shows significant improvements and the results now exhibit an excellent agreement with experimental data of Ronneberger.\(^5\) Moreover, the acoustic impedance \(Z\) of the cylindrical backward-facing step is also analyzed in the paper and is defined as follows:

\[
Z = \frac{p_d - p_u}{\rho cu'_u} = T_{12}.
\]

(5)
The definition can be directly derived from the transfer matrix notation (Eq. 4), since the acoustic pressure fluctuations upstream and downstream have to be equal due to the continuity condition at the sudden area expansion section. Thus, the transfer matrix element $T_{11}$ gives unity and the element $T_{12}$ simply represents the acoustic impedance (for further details see section III). For this reason, results are presented in terms of the acoustic transfer matrix (Eq. 4), whereas the acoustic scattering matrix (Eq. 1) has been used in publications.\textsuperscript{4, 6, 7, 13, 19} The results for the acoustic impedance are physically interpreted, compared with several analytical models\textsuperscript{20–23} and discussed in detail. Therefore, we verified that the LES/SI method is able to capture all relevant physical mechanisms between acoustics and the turbulent flow field.

Figure 1. Geometry of the cylindrical backward-facing step with contour plots of instantaneous axial velocity

III. The LES/SI Methodology

The numerical determination of the acoustic transfer behavior consists of a two step process: a compressible LES of the acoustic element under consideration, followed by a system identification process based on recorded, acoustic CFD data. In this case, the Wiener-Hopf-Inversion method is applied.\textsuperscript{24} Figure 2 illustrates the key steps of the procedure.

Figure 2. Fundamental approach of the identification of acoustic transfer matrices by applying the Wiener-Hopf-Inversion method based on LES raw data
Firstly, the LES is set up. The flow solver AVBP developed by CERFACS\textsuperscript{a} is applied.\textsuperscript{25} After having obtained a statistically stabilized solution, a transient simulation is conducted, in which broadband low-amplitude plane waves are superimposed on the mean flow at the in- and outflow boundaries of the computational domain. The temporal variations of the wave amplitudes $\partial f/\partial t$ at the inlet condition and $\partial g/\partial t$ at the outlet condition are chosen as the excitation signals, according to Poinsoet and Lele.\textsuperscript{26} Hence, during the simulation, acoustic waves are travelling through the CFD domain simultaneously in the up- and downstream direction, while being reflected and transmitted at the sudden change in cross-section. The generation of vortices at the trailing edge of the area change in response to impinging acoustic waves – a significant mechanism for dissipation of acoustic energy\textsuperscript{6} – is also considered.

For the system identification, the ingoing characteristic wave amplitudes $f_u$ and $g_d$ are considered as signals $s$, whereas the outgoing characteristic wave amplitudes $f_d$ and $g_u$ are considered as responses $r$. In the URANS simulations, these signals and responses can be easily retrieved by subtracting the mean values of flow variables from the instantaneous values. On the other hand, the resolved turbulence fluctuations of an LES make it difficult to differentiate between acoustic and turbulent fluctuations of pressure and velocity. In order to extract acoustic signals from LES data, a characteristics based filter (CBF) was developed by Kopitz et al.\textsuperscript{27} For the identification of the acoustic plane wave component, the CBF utilizes the fundamental effect that the acoustic waves propagate with the speed of sound, whereas the turbulent fluctuations are convected with the order of the average flow velocity.

The accuracy of the identification procedure described above depends strongly on the reflection coefficients at the in- and outlet boundary conditions, as shown by Yuen et al.\textsuperscript{28} If the reflection coefficients are too large, the ingoing characteristic wave amplitudes $f_u$ and $g_d$ will be correlated to some extent, which results in a significant error in the inversion of the Wiener-Hopf equation (the auto-correlation matrix $\Gamma$ becomes ill-conditioned). Hence, non-reflecting boundary conditions are applied, ensuring that the acoustic reflections are kept at the lowest possible level. In the present simulations, a modified version of the characteristics-based boundary conditions (NSCBCs) of Poinsoet and Lele\textsuperscript{26} was used. As stated by Polifke et al.,\textsuperscript{29} the appearance of partial reflection at low frequencies makes it necessary to extend the original NSCBCs formulation by a correction term. This corresponds to the acoustic plane wave amplitudes leaving the domain. In order to further improve the boundary treatment proposed by Polifke et al.,\textsuperscript{29} Kaess et al.\textsuperscript{30} introduced the plane wave "masking".\textsuperscript{31} Therefore, they coupled the NSCBCs with the CBF by utilizing the latter not only for extracting the acoustic signals, but also for the determination of the required correction term very accurately.

The post-processing is done by analyzing the recorded CFD time series of signals and responses (see Fig. 2, bottom). Therefore, the correlation based Wiener-Hopf-Inversion (WHI) was chosen to identify the acoustic transfer behavior of the simulated system.\textsuperscript{24,32} At first, the cross-correlation vector $c$ of signals $s$ and responses $r$ and the auto-correlation matrix $\Gamma$ of the signals $s$ are calculated. Afterwards, the Wiener-Hopf equation is set up and solved with respect to the unknown unit-impulse-response vector $h$. Then, a z-transformation of the latter is performed in order to obtain the scattering coefficients of the system in the frequency domain. Finally, the transfer coefficients are calculated with Eq. 1 - 3.

### IV. Numerical Setup

In this paper, the LES/SI method was used in order to determine the transfer coefficients of a cylindrical backward-facing step (Fig. 1). It consists of two co-axially connected pipes with diameters $d_u$ upstream and $d_d$ downstream. The upstream pipe has a length of $l_u$, whereas the length of the downstream pipe was set to $l_d$ (see Table. 1 for details). The flow enters the domain at the beginning of the upstream pipe and leaves the domain at the end of the downstream pipe. At the area expansion, flow separation occurs and a shear layer develops in the downstream pipe. In contrast to the simplicity of the geometry, a complex aeroacoustic interaction can be observed when the impinging acoustic waves directly interact with the unstable shear layer.\textsuperscript{5} The LES/SI approach should be able to capture this phenomena, since the methodology allows for the direct interaction of the acoustic waves and the resolved vortical structures. The area ratio $A_R = 0.35$ was selected because for this ratio experimental measurements are available.\textsuperscript{5} These measurements were utilized to validate the LES/SI results. For this ratio, the acoustical transfer behavior has been analyzed at different Mach numbers and ambient conditions which are summarized in Table. 1.

\footnote{http://www.cerfacs.fr/4-26334-The-AVBP-code.php}

American Institute of Aeronautics and Astronautics
Table 1. Geometrical information and operating conditions

<table>
<thead>
<tr>
<th>Variable name</th>
<th>Symbol</th>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>pipe diameter, upstream</td>
<td>$d_u$</td>
<td>-</td>
<td>0.05m</td>
</tr>
<tr>
<td>pipe diameter, downstream</td>
<td>$d_d$</td>
<td>-</td>
<td>0.085m</td>
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<td>pipe length, upstream</td>
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<td>-</td>
<td>0.5m</td>
</tr>
<tr>
<td>pipe length, downstream</td>
<td>$l_d$</td>
<td>-</td>
<td>0.7m</td>
</tr>
<tr>
<td>area ratio</td>
<td>$A_R$</td>
<td>$(d_u/d_d)^2$</td>
<td>0.35</td>
</tr>
<tr>
<td>pressure, ambient</td>
<td>$p_0$</td>
<td>-</td>
<td>101300Pa</td>
</tr>
<tr>
<td>temperature, ambient</td>
<td>$T_0$</td>
<td>-</td>
<td>298.15K</td>
</tr>
<tr>
<td>speed of sound, ambient</td>
<td>$c_0$</td>
<td>$\sqrt{\kappa R T_0}$</td>
<td>346.177m/s</td>
</tr>
<tr>
<td>Mach number, inlet</td>
<td>$M$</td>
<td>$u/c_0$</td>
<td>0.05...0.2</td>
</tr>
<tr>
<td>Reynolds number, inlet</td>
<td>$Re$</td>
<td>-</td>
<td>$4.7 \times 10^4...1.88 \times 10^5$</td>
</tr>
</tbody>
</table>

IV.A. Solver Settings and Mesh Quantities

The flow solver AVBP was used to perform the LES runs. It solves the compressible Navier-Stokes equations on unstructured grids based on a Finite Volume or Finite Element scheme, namely the Lax-Wendroff scheme or the Two-Step Taylor-Galerkin scheme. For the simulations, we selected the 2nd order Lax-Wendroff scheme for the spatial derivatives in combination with the explicit Euler time-stepping approach (2nd order) for the temporal discretization. The time step was adjusted in order to correspond to the Courant-Friedrich-Levy (CFL) number of 0.7. The effect of unresolved small-scale turbulent fluctuations on the large-scale resolved turbulent structures was modeled by the so-called Wall Attached Layer Eddy (WALE) model developed by Nicoud and Ducros. Compared to Smagorinsky type models, the WALE model is more preferable due to its superior ability in modelling the sub-grid-scale fluctuations of shear flows. Thus, latter model allows for a more accurate representation of the interactions between acoustic sound waves and coherent turbulent structures in the shear layer.

The computational grid consisted only of block structured hexahedral elements. The individual blocks formed an O-grid in the pipe cross-sections (radial $r$ and circumferential $\phi$ directions). In the axial direction, the elements were projected with an adapted grid spacing. Close to the backward-facing step, the mesh was consecutively refined in axial direction in order to resolve the initial development of the shear layer appropriately. Due to the different Re numbers for the analyzed cases, the overall mesh size needed to be adapted to the specific operating conditions. The average mesh resolution is summarized in Table 2.

Table 2. Averaged mesh resolution, normalized by wall units ($r$: radial direction, $\phi$: circumferential direction, $z$: axial direction)

<table>
<thead>
<tr>
<th>resolution in direction</th>
<th>M=0.05</th>
<th>M=0.1</th>
<th>M=0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^+$,$z^+$</td>
<td>55</td>
<td>70</td>
<td>120</td>
</tr>
<tr>
<td>$\varphi^+$</td>
<td>60</td>
<td>90</td>
<td>150</td>
</tr>
<tr>
<td>$r^+$ near-wall region</td>
<td>35</td>
<td>35</td>
<td>35</td>
</tr>
</tbody>
</table>

The wall boundary was described by an adiabatic no-slip condition. In combination with the mesh refinement close to the walls (Table 2), a two-layer wall function model was applied in order to compute the approximated wall shear stress velocity. The wall model switches between the linear law and the logarithmic law depending on $r^+$.

Beside the acoustic perturbations entering the domain at in- and outlet boundary, the convective conditions were defined as follows. At the inlet boundary, a fully turbulent velocity profile at the corresponding Re number was imposed. Turbulent fluctuations were omitted for the simulation results shown in this paper. At the outlet boundary, the static pressure at ambient conditions was imposed as averaged value on the outlet patch.
IV.B. Estimation of the Acoustic Dissipation and Dispersion Error

An important point to prove, is the ability of the solver AVBP to simulate accurately the propagation of acoustic waves with discretization schemes of 2nd order. Therefore, a validation study was carried out in the frequency range of interest, utilizing a simplified acoustic duct setup. The aim of the study was to determine the acoustic dissipation and dispersion error in dependence of the grid resolution and the numerical schemes selected.

The acoustic test setup consisted of a rectangular duct with equal height and width, but both much smaller than the axial extension of the duct. The length was adapted such that 10 complete periods of a 100Hz sinusoid at ambient conditions fit exactly into the duct. The walls were represented by the slip condition to avoid viscous losses. For the inlet and outlet boundary condition, the non-reflecting boundary conditions introduced in section II were applied. The computational grid consisted of equi-sized hexahedral elements. The cross-section of the duct was meshed with 3 cells in the corresponding directions, whereas in axial direction, the grid spacing was adjusted to yield explicitly a defined number of cells per wavelength of the 100Hz sinusoid. Its characteristic wave amplitude was set to $f_{\text{sin}} = 0.2m/s$. Thus, the flow field was initialized with a constant axial velocity of $u_{\text{ax}} = 0.25m/s$ in order to avoid the occurrence of flow reversal in case of negative acoustic velocity. 10 simulations were carried out, where the resolution varied in steps of 10 from 10 to 100 cells per wavelength. For the simulations, the 2nd order Lax-Wendroff scheme at a CFL number of 0.7 with explicit Euler time stepping was applied to ensure comparable results with the LES/SI scheme settings.

During the transient simulations, the sinusoidal forcing was imposed at the inlet boundary condition. Thus, the acoustic wave propagated from the inlet towards the outlet boundary. While the wave was travelling through the domain, time series of acoustic pressure and velocity were extracted at various positions along the duct. These time series were post-processed in order to determine the averaged acoustic dissipation and dispersion of the sinusoid. Therefore, the characteristic wave amplitude at each position was compared to the ingoing characteristic wave amplitude. Typically, an increasing reduction of the amplitude along the axial direction is observed if the acoustic wave is not properly resolved, i.e. for 10 cells per wavelength in Fig. 3).

![Figure 3. Dissipation of the acoustic wave amplitude plotted over 9.5 periods](image)

The loss in amplitude normalized by the reference amplitude was defined as dissipation error which is given in percent in Table 3. In case of wave dispersion, an under-resolved wave is mapped to a lower frequency compared to the reference frequency. This leads to an increasing wavelength, while the wave is travelling along the duct. The relative difference between calculated wavelength and the reference wavelength of the sinusoid was defined as dispersion error (Table. 3).
Table 3. Numerical dissipation and dispersion error after 10 periods

<table>
<thead>
<tr>
<th>cells per wavelength</th>
<th>dissipation error [%]</th>
<th>dispersion error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>90</td>
<td>0.06</td>
<td>0.02</td>
</tr>
<tr>
<td>80</td>
<td>0.08</td>
<td>0.03</td>
</tr>
<tr>
<td>70</td>
<td>0.14</td>
<td>0.02</td>
</tr>
<tr>
<td>60</td>
<td>0.18</td>
<td>0.04</td>
</tr>
<tr>
<td>50</td>
<td>0.23</td>
<td>0.04</td>
</tr>
<tr>
<td>40</td>
<td>0.40</td>
<td>0.05</td>
</tr>
<tr>
<td>30</td>
<td>1.15</td>
<td>0.05</td>
</tr>
<tr>
<td>20</td>
<td>2.77</td>
<td>0.08</td>
</tr>
<tr>
<td>10</td>
<td>34.71</td>
<td>2.89</td>
</tr>
</tbody>
</table>

In the following paragraph, a worst case estimation of the acoustic dissipation and dispersion error is calculated for the LES runs of the backward-facing step. Therefore, we begin with the determination of the minimal wavelength which has to be considered. Here, the upper frequency limit $f_{r, \text{max}}$ of the excitation signal is 5000 Hz, representing the highest frequency of interest. At ambient operating conditions, the corresponding wavelength $\lambda_{f_{r, \text{max}}}$ is given by the following expression.

$$\lambda_{f_{r, \text{max}}} = \frac{c_{0}}{f_{r, \text{max}}} = 0.0692m. \quad (6)$$

With the wavelength calculated and by taking into account the domain extension and the largest cell size in axial direction $\Delta z$, the number of wave periods and the number of cells per wavelength can be computed.

For the first quantity, the distance of the wave extraction positions $\Delta x_{\text{extr}}$ in the LES/SI runs equaled 0.75m. In conjunction with the wavelength $\lambda_{f_{r, \text{max}}}$, this yields 10.83 wave periods between the extraction positions for the highest frequency. For the second quantity, the coarsest grid spacing in $z$-direction $\Delta z_{\text{max}}$ was $1.19 \times 10^{-3}m$. Hence, the ratio of the wavelength $\lambda_{f_{r, \text{max}}}$ to the coarsest grid spacing $\Delta z_{\text{max}}$ results in 58 cells per wavelength.

The corresponding values of the maximal dissipation and dispersion error can now be extracted from Fig. 3 and Table. 3. The values are approximately 0.2% for the dissipation error and 0.04% for the dispersion error. Thus, it can be concluded that the grid spacing in combination with the selected discretization scheme and a CFL number of 0.7 ensures a high quality of the acoustic propagation behavior for the LES of the backward-facing step.

IV.C. Settings for Acoustic Signal Analysis

In this subsection, the relevant parameters for a successful system identification, which are specific for the analyzed system, are introduced and discussed. Namely the type of the excitation signal, its frequency content, the duration of the transient simulation, the CBF settings and the estimation of relevant time lags for the Wiener-Hopf-Inversion.

As mentioned above, broadband, low-amplitude plane waves were superimposed on the mean flow at both in- and outflow boundaries. Therefore, two statistically independent Pseudo Random Binary Signals (PRBS) were generated. They were frequency band-limited from 0 - 5000 Hz to avoid excitation of the first high order mode of the downstream pipe. In order to ensure statistically converged results of the acoustic signal analysis, the flow time $\Delta T_{\text{sim}}$, last 0.2 s. This represents 92 times the acoustic time scale of the configuration, which is given by $T_{\text{ac}} = \Delta x_{\text{extr}} / c_{0} = 2.17 \times 10^{-3}s$.

The quality of the CBF was analyzed by Kopitz et al. Moreover, the relative accuracy of the extracted wave amplitudes was determined in dependency of the number of planes and distance between them. For the simulations here, ten monitor planes upstream and downstream, respectively, were found to be sufficient to filter out the characteristic wave amplitudes from the instantaneous flow variables. The distance $\Delta x$ between the CBF planes was 0.02m. The reference monitor planes, at which the filtered characteristic wave amplitudes from the ten monitor planes were computed, were located 0.25m upstream and 0.5m downstream.
of the discontinuity. For the adjacent post-processing, it was sufficient to extract the characteristic wave amplitudes every 10th simulation time step. This resulted in a time interval $\Delta t_{\text{extr}}$ of $7.5 \times 10^{-6}$s for $M = 0.05$ and of $5 \times 10^{-6}$s for $M = 0.1$ and $M = 0.2$.

Since the Wiener-Hopf-Inversion determines the unit-impulse-response (UIR) of a linear time, invariant (LTI) system, the most important parameters is the filter length $L$ of the UIR. The latter value determines the longest time lag of the analyzed LTI system which can be captured by the UIR. In this study, the memory time of the UIR was set to $t_{\text{UIR}} = L \Delta t_{\text{extr}} = 3.75 \times 10^{-3}$ s, which covers the wave propagation and possible sound wave shear layer interactions.

V. Results

The result section is subdivided into two parts. At first, the results for the transfer matrix coefficients are introduced and discussed. Secondly, the acoustic impedance is presented and the involved physical mechanisms are discussed in detail by comparing the LES/SI data with analytical models.

V.A. Transfer Matrix Coefficients

For a broad range of frequencies, the results of the transfer coefficients are shown in Fig. 4.

![Figure 4. Amplitudes of the transfer coefficients for three inlet Mach numbers, plotted over the upstream Helmholtz number and compared with experimental results of Ronneberger](image-url)
The amplitudes of the coefficients are plotted versus the upstream Helmholtz number

\[ \text{He}_u = \frac{\pi f_r d_u}{c_0}, \]  

(7)

with \( f_r \) representing the frequency of the acoustic waves.

In general, the excellent agreement of the LES/SI results with the experimental data\(^5\) demonstrates clearly that the LES/SI approach can be applied to characterize accurately the transfer behavior of the cylindrical backward-facing step in the linear acoustic regime.

By analyzing the coefficients individually, the results for the element T11 and T22 match the expectations, since they describe how the acoustic pressure \( p' \) or velocity \( u' \) of the upstream and downstream pipe are associated with each other (Eq. 4). In case of the T22 coefficient, the acoustic mass flux is constant. Since the transfer matrix notation is based on the acoustic velocity \( u' \), the amplitude of T22 indicates the area ratio of the backward-facing step (Table 1). Moreover, the phase angle behavior of these two coefficients, which is not shown in this paper, is necessarily zero for the entire frequency range and all Mach numbers.

The other two coefficients T12 and T21 in Fig. 4 represent the relation between the upstream acoustic pressure and the downstream acoustic velocity and vice versa. For simple pipe elements, the amplitude of these coefficients would be zero and the phase angle would be equal to \( \pm \frac{\pi}{2} \). In case of the area expansion, the coefficients show the pipe end-corrections, which are typically interpreted in terms of effective lengths. The amplitude behavior of the T21 coefficients depicts only small values for the effective length and indicates only a slight dependence on the Mach number. Furthermore, the phase angle behavior, which is not shown here, equals \( \pm \frac{\pi}{2} \). In contrast, the amplitude of the T12 coefficient, namely the acoustic impedance, shows a complex behavior depending on frequency and Mach number. For this reason, it is analyzed in more detail in the next subsection.

V.B. Acoustic Impedance

In the following, the acoustic impedance of the cylindrical backward-facing step (Fig. 1) is analyzed for different Mach numbers and a broad range of frequencies up to the cut-off frequency of the first high order mode of the downstream pipe. In literature, a so-called “lumped” impedance is often derived analytically to describe the acoustic transfer behavior in case of a sudden area expansion or contraction.\(^{20–23}\)

This impedance is a complex valued quantity

\[ Z = T_{12} = \frac{p'_d - p'_u}{\rho c u'_u} = R_{\text{resistance}} + i \omega l_{\text{reactance}}. \]

(8)

Its real part \( R \) is called the acoustic resistance and contains the loss of acoustic energy whereas the imaginary part \( \omega l_{\text{reactance}} \) is called the acoustic reactance and describes the temporary storage of acoustic energy in the evanescent high order modes. Since both parts of the impedance depend on the mean flow velocity and the frequency of the ingoing acoustic wave, it is impossible to derive analytical expressions without simplifications and assumptions, which would in turn lead to less accurate acoustic models. Thus, the LES/SI approach was applied to obtain accurate results for both acoustic resistance and reactance. The necessary information were extracted from the transfer matrix element \( T_{12} \). The impedance behavior is evaluated and compared with the available analytical models.\(^{20–23}\) Moreover, its dependency on frequency and Mach number is discussed in more detail.

Before the impedance behavior is discussed, the analytical models are introduced. Here, we restrict ourselves to a brief recapitulation of these methods. The philosophy of the derivation is sketched to highlight the applicability limits of each model. Details of the models are presented in the corresponding publications.

Karal’s model\(^{20}\) and Peat’s model\(^{22}\) belong to the class of so-called “mode-matching” models. In order to derive these models, the convective wave equation or the Helmholtz equation for the acoustic pressure field is solved. The solution of latter equations describes the acoustic pressure field as a linear superposition of the plane wave mode and an infinite number of high order modes. In addition, the acoustic velocity field is expressed in terms of a multimodal expansion of admittances, which are derived from the Euler equation in the axial direction. The general solution of the mode shapes for the acoustic pressure and velocity field is determined separately for each pipe. In a next step, the continuity condition for the acoustic pressure and
velocity field at the area expansion is introduced to obtain the combined solution of both pipes. In other words, the mode shapes of the up- and downstream pipe have to match or coincide at the discontinuity. After having derived an equation system with an infinite number of equation, an individual equation for each mode shape, the solutions of the acoustic pressure and velocity amplitudes can be calculated. In practical cases, only the first 30 modes need to be considered, since the influence of the high order modes on the impedance behavior decreases rapidly with increasing mode order. In a final step, the iteratively computed solutions of the mode amplitudes at the discontinuity are inserted into the impedance definition (Eq. 8). Here, it is crucial to point out that the reactance contains the effects of the evanescent high order modes generated through the scattering process at the area expansion. Their influence can be interpreted as a phase shift of the plane wave mode.

The different assumptions and simplifications of Karal’s and Peat’s models are introduced in the following paragraph. From the differences of the two models, the physical effects, which are captured by the individual model, can be clearly highlighted.

First of all, Karal’s model is based on the solution of the Helmholtz equation and therefore, does not consider mean flow effects. Furthermore, the admittances are frequency independent due to simplifications. This means that the damping factors for the high order modes are only accurate for the low frequency regime. Moreover, he neglected a term with minor influence in the final equation for the acoustic impedance. In contrast to Karal, Peat’s model is based on the convective wave equation, which includes 1-dimensional mean flow effects. Thus, the model is able to predict the loss of acoustic energy due to the total pressure drop over the area change. The admittances are frequency dependent, which means that the damping corrections for the high frequency range are considered. In total, this results in an accurate analytical model including the high frequency behavior.

Auregan et al. utilized a simplified mode-matching model. Their model is based on the 2-dimensional Euler equation for the admittances and does not include mean flow effects. Therefore, the impedance consists only of an imaginary component, since mean flow effects are not taken into account. Compared to the previous models of Karal and Peat, this model computes only the impedances and admittances for the plane wave mode and a single equivalent high order mode. The latter mode should not be mixed up with a physical mode propagating in the pipes, since it describes the effect of all high order modes utilizing a single mode shape for the acoustic pressure and velocity field in combination with an associated wave number. Thus, an approximated velocity profile needs to be introduced in order to allow for the computation of the equivalent admittance. Auregan et al. derived a simplified equation which reconstructs the radial pressure field at the discontinuity quite successfully. Hence, the high frequency damping effects on waves just below the cut-off frequency should be well approximated.

The last analytical model, which is used in order to determine the “lumped” impedance, was derived by Morse and Ingard. Here, the authors basically assumed that the sound waves propagate along the stream lines of the potential flow solution for the analyzed configuration. This assumption is valid for 2-dimensional flow problems with negligible mean flow. In case of simple duct flows, the stream lines are oriented in the same direction, meaning that only the plane wave mode exists. Close to flow constrictions, like orifices and sudden area changes, the potential stream lines are not aligned with each other and thus, the plane wave mode is deformed and partly scattered into a high order mode. A conformal mapping method, the Schwartz-Christoffel transformation, is applied to transform the potential flow solution in such cases into a flow solution with aligned stream lines, like for potential flows in ducts. Then, the acoustic energy is integrated over the transformed element, which gives finally the acoustic resistance and reactance for the discontinuity.

Before latter model can be compared with the other solutions mentioned above, the results of Morse’s and Ingard’s model need to be transformed from a 2-D rectangular geometry into a cylindrical configuration. For this purpose, the Helmholtz scaling approach from Boij and Nilsson is utilized. The equation below rescales the frequency based on the normalized Helmholtz number He:

\[
\text{He}^* = \frac{1}{\pi} \frac{(k_0 h_u)_{rec}}{A_R} = \frac{1}{\pi} \frac{(k_0 r_u)_{cyl}}{\kappa_0}.
\]

Here, \((k_0 h_u)_{rec}\) and \((k_0 r_u)_{cyl}\) represent the Helmholtz numbers of the rectangular and the cylindrical geometry. Moreover, \(\pi\) and \(\kappa_0 \approx 3.832\) describe the cut-off Helmholtz numbers of the first higher-order mode of a rectangular duct and a cylindrical pipe, respectively.

After the introduction of the analytical models and their main features, we are able to proceed with
the discussion of the acoustic impedance and its dependency on acoustic flow field interaction and wave scattering characteristics. The impedance is plotted with respect to its resistance and reactance components (Eq. 8) in Fig. 5 and 6. Furthermore, the amplitude and phase angle notation is utilized in order to discuss the impedance behavior (Fig. 7 and 8). For a clear representation of the results, Fig. 6 and 7 are split into two subplots, namely (a) and (b). Subplots (a) always show the numerical and experimental results for the lowest Mach number in comparison to analytical models which do not consider mean flow effects, e.g. Morse and Ingard,21 Auregan et al.,23 and Karal.20 In subplots (b), the impact of the Mach number on the impedance is presented by plotting results of Peat’s model, the experiments and the LES/SI method.

![Figure 5. Acoustic resistance plotted over the upstream Helmholtz number; comparison of LES/SI results, experimental data of Ronneberger5 and the analytical model of Peat22 for three inlet Mach numbers](image)

![Figure 6. Acoustic reactance plotted over the upstream Helmholtz number: a) comparison of LES/SI result and experimental data of Ronneberger5 for M = 0.05 with models of Karal,20 Auregan et al.,23 Morse and Ingard21 and Peat,22 b) comparison of LES/SI result for three inlet Mach numbers with the Mach number dependent model of Peat22 and experimental results of Ronneberger5](image)

It makes sense to begin the discussion with the resistance and reactance plots (Fig. 5 and 6), because the important features of the impedance behavior are clearly visible if this plotting notation is utilized. By analyzing Fig. 5 and 6, three frequency ranges with different impedance characteristics can be distinguished from each other. The first range covers the low Helmholtz number regime from 0 to values between 0.2 and 0.5 which depends on the mean flow velocity (see Fig. 5). It is followed by an intermediate range which ends at He ≈ 1. In this range, we observe constant resistance levels (Fig. 5) and a constant slope for the reactance (Fig. 6). The third range characterizes the impedance at high Helmholtz numbers up to the cut-off
frequency of the first high order mode (He ≈ 2.2) (Fig. 6).

The three different frequency ranges are associated with three different physical effects which lead to the observed impedance behavior.

The first one, represents the well-known end-correction at pipe discontinuities. It can be observed at the low and intermediate Helmholtz number ranges as constant slope for the reactance, when the Mach number is very low (Fig. 6a). The constant slope can also be identified for higher Mach numbers in Fig. 6b. But only the intermediate range shows this behavior, whereas the reactance remains constant in the low Helmholtz number regime, i.e. He < 0.5 for M = 0.2.

This observation leads directly to the second physical mechanism influencing the impedance behavior at small Helmholtz numbers. The resistance plot (Fig. 5) indicates increasing values with increasing Helmholtz number. The most interesting fact is here, that the resistance is constant at the beginning. This regime is followed by a frequency range at which the resistance gradually changes to a higher level. At the end of this range, the resistance becomes constant again. The described behavior is associated with the Mach number of the flow in the upstream pipe and is best observed for the cases M = 0.1 and M = 0.2. A similar behavior for the area expansion case were already reported by Boij and Nilsson for the upstream reflection coefficient of the scattering matrix. They associated their observations with the hydrodynamic instability in the vicinity of the shear layer, which is triggered by the ingoing acoustic waves for a certain Strouhal number range. This interaction process leads to an energy transfer from the acoustic field to the flow field which results in the loss of acoustic energy or an increase of resistance for the results shown in this paper. Ronneberger’s measurements are in good agreement with the LES/SI data, verifying the existence of this phenomena. In contrast, Peat’s model is not able to capture this effect, since the model was derived from the convective wave equation. Hence, it does not contain aeroacoustic interactions. As mentioned above, the interaction becomes stronger if the Mach number increases. Thus, the model and the LES/SI results deviate more from each other with increasing Mach number in Fig. 5 and 6b.

The third physical mechanism only affects the acoustic scattering process of high frequencies below the cut-off frequency of the first high order mode. In Fig. 6a-b, the increasing reactance is caused by a delayed back-scattering of acoustic energy from the high order modes into the plane wave mode, since the admittances of the high frequency non-planar modes decrease with increasing frequency. This leads to longer propagation distances until the high order modes vanish completely. This effect is clearly shown by the analytical models in Fig. 6a. Karal’s model and the one of Morse and Ingard neglect the frequency dependency of the high order mode admittances, whereas the models of Peat and Auregan et al. take the dependency into account.

In addition to the resistance and reactance behavior, it is interesting to analyze the impact of the physical phenomena on the acoustic impedance by utilizing the amplitude and phase angle notation (see Fig. 7 and 8). The three Helmholtz number ranges exist also in these plots and the discussion is following the different regimes as before.

In the low Helmholtz limit, the impedance models of Auregan et al., Karal, and Morse and Ingard tend to zero, since these models take only the wave scattering effect at the discontinuity into account. In contrast, the model of Peat and the LES/SI method predict values of 0.04 at the lowest frequency. This behavior can be expected, because both approaches consider the loss mechanism of acoustic energy due to the total pressure drop over the discontinuity. Hence, the real part of the impedance (Fig. 5) leads to an amplitude deviation from zero and the values predicted by the other models. Furthermore, the curves of the LES/SI method and Peat’s model begin with a small slope, which becomes stronger with increasing Helmholtz number. At He = 0.2, the slope of latter two results reaches the same value as described by the other analytical models. The only exception is the model of Morse and Ingard, which predicts a slightly higher slope. This may be explained by the improper assumption of modelling the acoustic mode shape based on the potential flow solution for the configuration. Another important fact is given by the constant offset of different models and the LES/SI results in Fig. 7a. In the low Helmholtz range, the curves do not coincide perfectly, because different levels of resistance are calculated by the individual models (see Fig. 5).

In the intermediate Helmholtz range, the slope of all curves remains constant, which indicates that the effective length is independent of frequency up to He ≈ 1. More precisely, the resistance is constant and the reactance depends linearly on the frequency as it can be observed in the plots for the resistance (Fig. 5, M=0.05) and the reactance (Fig. 6a). Moreover, the majority of Ronneberger’s results lies in this range and show an excellent agreement with the models and the LES/SI result.

Beyond He = 1, the predictions by models of Peat and Auregan et al. and the LES/SI data describe
Figure 7. Amplitudes of the acoustic impedance plotted over the upstream Helmholtz number: a) comparison of LES/SI results and experimental data of Ronneberger\(^5\) with models of Karal,\(^{20}\) Auregan et al.,\(^{23}\) Morse and Ingard\(^{21}\) and Peat\(^{22}\) for \(M = 0.05\), b) comparison of LES/SI results for three different inlet Mach numbers with Peat’s model\(^{22}\) and experimental results of Ronneberger\(^5\).

A quadratic increase of the amplitudes with frequency, whereas all low Helmholtz models fail to predict this increase. The increase is supported by Ronneberger’s measurements at least up to \(\text{He} = 1.3\), the highest measured frequency.\(^5\) The model of Auregan et al. shows a smaller curvature leading to significant deviation above \(\text{He} = 1.9\). This possibly follows from the assumed velocity profile of the equivalent high order mode.\(^{23}\) The maximal amplitude is reached at the cut-off of the first high order mode (\(\text{He} = 2.2\)).

The impact of the mean flow on the impedance is visualized in Fig. 7b. Here, only the model of Peat\(^{22}\) and the LES/SI data are compared with the experimental data.\(^5\) The three ranges and their associated physical effects can also be identified in this plot. Both, LES/SI results and Peat’s model\(^{22}\) show higher amplitude levels for low Helmholtz numbers, if the mean flow velocity increases. Especially, the LES/SI results show an interesting behavior when \(\text{He} = 0.3\) is reached. There, all curves coincide between Helmholtz numbers of 0.3 and 0.6 indicating that the amplitudes are Mach number independent. This behavior is associated with the sound wave shear layer interaction discussed above. The analytical model of Peat\(^{22}\) gives fairly good results for \(M = 0.05\) and \(M = 0.1\). But a deviation is observed for \(M = 0.2\) at low Helmholtz numbers, which can be expected, because the model is limited to incompressible, low Mach number flows as mentioned by Peat\(^{22}\).

Figure 8. Phase angle of the acoustic impedance plotted over the upstream Helmholtz number; comparison of LES/SI results and experimental data of Ronneberger\(^5\) with analytical models of Karal,\(^{20}\) Auregan et al.,\(^{23}\) Morse and Ingard\(^{21}\) and Peat\(^{22}\) for three inlet Mach numbers (black: \(M = 0.05\), red: \(M = 0.1\), blue: \(M = 0.2\)).
Analogous to the amplitudes, the phase behavior can be interpreted (Fig. 8). Beginning with the analytical models without mean flow,\textsuperscript{20, 21, 23} their phase angle remains constant at $-\pi/2$ (resistance equals zero). The model of Peat\textsuperscript{22} shows also the expected behavior. All curves begin at zero in the low Helmholtz number limit and tend towards $-\pi/2$ with increasing frequency. The LES/SI results show a similar behavior as Peat’s data, but deviate from latter model in the low frequency range and close to the cut-off frequency ($He \approx 2$). In both cases, the sound vortex interaction is responsible for the different predicted values. This becomes clear if one compares the behavior with the resistance and reactance in these ranges (Fig. 5 and 6b). In the low frequency range, the LES/SI method shows constantly small negative values for the reactance, which is in contradiction to the model of Peat.\textsuperscript{22} Thus, LES/SI results do not tend to zero for the phase angle if the Helmholtz number approaches zero. Moreover, the values of the resistance for high Helmholtz and Mach numbers decrease when the frequency is close to the cut-off condition (Fig. 5). Therefore, the phase angle of the LES/SI results equals $-\pi/2$.

VI. Conclusions

An efficient hybrid approach of broadband excited LES combined with system identification methods makes it possible to study aeroacoustic interactions in turbulent wall-bounded flows. The aeroacoustic LES is rather expensive compared to other numerical aeroacoustic methods, but it allows for direct interactions of sound waves and coherent turbulent structures which cannot be reproduced by simplified numerical methods. Here, the efficiency of the approach clearly benefits from the high parallelization capabilities of the flow solver AVBP, which exhibits an almost ideal speedup on high performance cluster systems. In combination with broadband identification, it is possible to determine acoustic transfer coefficients for a relevant range of frequencies within a few days.

The numerical accuracy of the second order scheme in combination with the mesh resolution and the selected CFL number was validated by acoustic test simulations. The propagation characteristics in terms of the dissipation and dispersion error were evaluated. In case of the LES runs for the backward-facing step, the analysis indicated negligible errors.

The results obtained for the transfer coefficients of plane acoustic waves at a sudden change of cross section in a pipe exhibit excellent agreement with experimental data. The observed behavior for the majority of the coefficients match the expectations and is governed by the continuity condition for the acoustic pressure and velocity at the pipe discontinuity. The coefficient, which represents the acoustic impedance from the upstream to the downstream pipe, shows a complex and distinct dependency on the mean flow velocity and the frequency. The comparison of this coefficient with several analytical models gave further insights into the involved mechanisms and proved the applicability of the LES/SI method to capture different aeroacoustic effects. The analytical solutions were selected based on their ability to model the different effects.

The LES/SI method considers the scattering of the plane wave mode into high order modes, which is interpreted as end-correction or effective length. In the high frequency regime close to the cut-off of the first high order mode, the delayed back-scattering of acoustic energy from the evanescent high order modes into the plane wave mode is also predicted correctly. For latter two effects, the results are in accordance with the analytical models. The aeroacoustic interaction – a triggered instability of the shear layer by impinging waves from upstream in a certain frequency range – is not captured by any of the models utilized here except the compressible LES. The correct prediction of the interaction with the LES/SI method is validated by the experimental data.

The mentioned benefits and the excellent accuracy make the proposed approach very attractive for the identification of acoustic elements in the linear regime, which involve complex aeroacoustic interactions.

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