A nonlinear model for the air-breathing flexible hypersonic vehicle (HSV) is provided giving consideration to the complex interactions between the propulsion system, aerodynamics, and structural dynamics. This nonlinear HSV is then approximated by a set of linear models over a large operating envelop, and a controller is synthesized using $H_{\infty}$ Linear Parameter Varying (LPV) techniques. Velocity tracking is simulated using a digital switching algorithm. A least squares optimization is performed on the tracking error state when switching between controllers to minimize the control effort needed by the actuators. The result is that velocity can be successfully tracked for this model over a range from Mach 7 to Mach 9, and an altitude range from 80,000 feet to 90,000 feet.

**Nomenclature**

- $c$ = coefficient of damping for HSV
- $D$ = drag force of the HSV
- $\tilde{\mathbf{F}}$ = force vector of the HSV
- $\tilde{\mathbf{F}}_b$ = force vector of the HSV in the body frame
- $g$ = acceleration due to gravity
- $h$ = altitude of the HSV
- $L$ = lift force of the HSV
- $M$ = moment of the HSV
- $m$ = mass of the HSV
- $N_n(t)$ = the generalized modal force acting on the HSV as a function of time
- $N_k(\rho)$ = bases of the null spaces of $[B^T_L \ D^T_L]$ 
- $P(t)$ = concentrated force acting on the HSV
- $\dot{p}$ = momentum vector of the HSV
- $p(x,t)$ = distributed force acting on the HSV
- $Q$ = pitch rate of the HSV
- $T$ = thrust force of the HSV
- $\tilde{\mathbf{V}}$ = velocity vector of the HSV
- $V_t$ = true airspeed of the HSV
- $w(x,t)$ = beam displacement as a function of time and distance along the x-axis of the beam
- $\alpha$ = angle of attack of the HSV
- $\gamma$ = $H_{\infty}$ performance index
- $\theta$ = pitch angle of the HSV
- $\rho$ = LPV scheduling parameter
- $\omega$ = natural frequency
- $\dot{\omega}$ = angular velocity vector of the HSV

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This work was partially supported by NASA under Grant NNX07AC40A.
I. Introduction

Research into air-breathing hypersonic vehicles started in the 1960’s. The purpose for these hypersonic vehicle research is to make space travel more affordable. Research in hypersonic vehicles continued through the 1990’s with the National Aerospace Plane. Since then, it has been recognized that advances in propulsion systems and materials will be needed to make this a feasible full-scale option for space travel. NASA has recently had a successful test of the scramjet powered X-43A. This is a sub-scale model of a hypersonic vehicle that flew in 2004 and 2005. The X-43A has had successful flight tests with a recorded speed of Mach 9.6.

Hypersonic vehicles require a highly integrated design approach. The vehicles typically consist of an airframe with an integrated scramjet propulsion system. The airframe itself is designed such that the bow shock acts as a compression stage to the scramjet itself. Often, there is a cowl door that can be adjusted to ensure that the bow shock is focused right on the tip of the entrance to the scramjet. For the purpose of this study, it is assumed that the cowl door will remain fixed. The aft of the airframe is designed such that it acts as a diffuser allowing the exhaust from the scramjet to expand externally. The vehicle geometry can be seen in Fig. 1 and Fig. 2. The cowl door in this figure is fixed, leaving four control efforts. These control efforts are the elevator, canard, fuel equivalence ratio, and the diffuser area ratio.

II. Aerodynamics and Equations of Motion

The main focus of this section is to use the aerodynamic model of a hypersonic vehicle as defined by Bolender and Doman\(^1\) to obtain a nonlinear plant model from which a \(H_\infty\) linear parameter varying (LPV) controller will be synthesized. A minimal amount of effort will be spent discussing the aerodynamics of the system, so it is recommended that the readers refer to the reference [1] for further information.

It is important to first understand the importance of including the flexibility of the vehicle into the equations of motion. The vibration of the hypersonic vehicle has an effect on the angle of the bow shock. As a result, this has an effect on the pressures downstream of the shock as well. These changes in pressure have an effect on the performance of the scramjet. Often times due to the shock, the scramjet will be operating outside of its optimal design region. This of course affects the thrust and moment of the hypersonic vehicle. It is therefore important to take the vibrational effects of the vehicle into consideration when modeling the equations of motion.

The particular hypersonic vehicle model used in this study incorporates a flexible body model. The following partial differential equation defines the vibration of the vehicle with respect to time and space.
Assume a solution in the form of

\[ w(x, t) = \phi(x)\eta(t) \]  

Using separation of variables, Eq. (1) can be separated into two ordinary differential equations:

\[ EI \frac{d^4\phi(x)}{dx^4} - \omega^2\dot{m}\phi(x) = 0 \]  

\[ \frac{d^2\eta(t)}{dt^2} + \frac{c}{\dot{m}}\eta(t) + \omega^2\eta(t) = 0 \]

Let,

\[ \beta^4 = \frac{\omega^2\dot{m}}{EI} \]  

Applying Eq. (5), Eq. (2) now becomes:

\[ \frac{d^4\phi(x)}{dx^4} - \beta^4\phi(x) = 0 \]

The solution to Eq. (6) is:

\[ \phi(x) = A \sin(\beta x) + B \cos(\beta x) + C \sinh(\beta x) + D \cosh(\beta x) \]  

For the free-free beam problem, the following boundary conditions are applied.

\[ \phi''(0) = \phi'''(0) = \phi''(L) = \phi'''(L) = 0 \]  

Applying the boundary conditions yields the following transcendental equation.

\[ \cos(\beta L) \cosh(\beta L) - 1 = 0 \]

There are an infinite number of solutions \( \beta_n \) that satisfy Eq. (9). Using modal analysis yields:

\[ \phi_n(x) = \left[ (\sin(\beta_n L) + \sinh(\beta_n L)) \cdot (\cos(\beta_n L) + \cosh(\beta_n L)) \\
+ (\cos(\beta_n L) - \cosh(\beta_n L)) \cdot (\sin(\beta_n L) + \sinh(\beta_n L)) \right] \]

Once \( \phi_n(x) \) is solved for, it is necessary to solve for \( \eta_n(t) \). Using a distributed and concentrated force, Eq. (1) becomes:

\[ EI \frac{d^4w(x, t)}{dx^4} + \dot{m} \frac{d^2w(x, t)}{dt^2} + c \frac{dw(x, t)}{dt} = p(x, t) + P_j(t)\delta(x - x_j) \]
The modal solution now takes the form:

\[ w(x, t) = \sum_{n=1}^{\infty} \phi_n(x)\eta_n(t) \]  

Looking specifically at \( \eta_n(t) \), the differential equation becomes,

\[ \ddot{\eta}_n + 2\xi \omega_n \dot{\eta}_n + \omega_n^2 \eta_n = N_n(t) \]  

where \( N_n(t) \) is the generalized modal force. This modal force is defined as,

\[ N_n(t) = \int_0^L \phi_n(x) p(x, t) dx + \sum_{j=1}^{l} \phi_n(x_j) P_j(t) \]  

where \( l \) is the number of concentrated forces applied to the beam. For this study, only the first three modes of vibration were considered.

Now that a model of the vibration of the hypersonic vehicle has been determined, it is important to look at the governing equations of motion for the vehicle. There are a number of assumptions that will be made to simplify the problem for this study. Only the longitudinal motion of the vehicle will be considered. It is assumed that there is no side slip, no lateral motion, and no roll for the hypersonic vehicle. This study will also use the flat earth assumption (i.e. the curvature of the earth will be neglected). Additionally, it will be assumed that the vehicle has a constant mass, and that the thrust from the scramjet is axial in the body frame.

Starting with the free body diagram shown in Fig. 3, and the momentum of the vehicle gives,

\[ \ddot{p} = m\ddot{V} \]  

Taking the time derivative of Eq. (15) yields:

\[ \dddot{F} = \frac{d\dddot{p}}{dt} = m\frac{d\dddot{V}}{dt} = m\ddot{a} \]  

Taking Eq. (16) and putting it into the body frame gives the following:
\[ \ddot{F}_b = \left( \frac{d\ddot{p}}{dt} \right)_b + \ddot{\omega} \times \dddot{p} = m \left( \frac{d\ddot{V}}{dt} \right)_b + m\ddot{\omega} \times \dddot{V} \]  
\quad (17)

where,
\[ \frac{d\ddot{V}}{dt} = \begin{bmatrix} \frac{du}{dt} \\
\frac{dv}{dt} \\
\frac{dw}{dt} \end{bmatrix} = \begin{bmatrix} \dddot{u} \\
\dddot{v} \\
\dddot{w} \end{bmatrix} \]
\quad (18)

Simplifying Eq. (17) by substituting Eq. (18) yields,
\[ \ddot{F}_b = m \begin{bmatrix} \dddot{u} \\
\dddot{v} \\
\dddot{w} \end{bmatrix} + m \begin{bmatrix} \dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3 \end{bmatrix} \begin{bmatrix} P \\
Q \\
R \end{bmatrix} = m \begin{bmatrix} \dddot{u} \\
\dddot{v} \\
\dddot{w} \end{bmatrix} + m \begin{bmatrix} Qw - Rv \\
Ru - Pw \\
 Pv - Qu \end{bmatrix} \]
\quad (19)

which can be rewritten as,
\[ \ddot{F}_b = \begin{bmatrix} F_x \\
F_y \\
F_z \end{bmatrix} = m \begin{bmatrix} \dddot{u} \\
\dddot{v} \\
\dddot{w} \end{bmatrix} + m \begin{bmatrix} Qw - Rv \\
Ru - Pw \\
 Pv - Qu \end{bmatrix} \]
\quad (20)

From the assumptions made for the problem, \( \ddot{v} = \nu = P = R = 0 \). These assumptions reduce Eq. (20) to be as follows.
\[ \ddot{F}_b = m \begin{bmatrix} \dddot{u} \\
0 \\
0 \end{bmatrix} + m \begin{bmatrix} Qw \\
0 \\
-Qu \end{bmatrix} \]
\quad (21)

Summing the forces and substituting yields the following equations of motion for the HSV in terms of lift, thrust, and drag.
\[ \dot{V}_t = \frac{(T \cdot \cos(\alpha) - D)}{m} - g \cdot \sin(\theta - \alpha) \]
\quad (22)
\[ \dot{\alpha} = \frac{(L + T \cdot \sin(\alpha))}{m \cdot V_t} + \frac{g \cdot \cos(\theta - \alpha)}{V_t} \]
\quad (23)
\[ \dot{Q} = \frac{M}{l_{yy}} \]
\quad (24)
\[ \dot{h} = V_t \cdot \sin(\theta - \alpha) \]
\quad (25)
\[ \dot{\theta} = Q \]
\quad (26)
Equations (22-29) are the governing equations of motion that will be used for this study. For a more detailed explanation of the derivation, see Bolender and Doman’s paper\(^1,9,5,4,7\). The equations of motion are listed in terms of moment (M), thrust (T), lift (L), and drag (D) on the vehicle. These values will vary depending upon the vehicle configuration, and the flight dynamics themselves. The calculations for these values can also be found in the Bolender and Doman paper\(^1,9\), but will not be discussed in this paper.

### III. LPV Control Techniques

The controller design for this vehicle will utilize \( H_\infty \)
linear parameter varying (LPV) control techniques. Therefore, it will be necessary for the reader to have an understanding of \( H_\infty \) control design to understand this report. This section will discuss the main concepts involved with implementing LPV controls to a HSV\(^2,3\).

From the previous section, it can be seen that the equations of motion are nonlinear. LPV control design process will take these nonlinear equations and linearize them at different operating conditions, or scheduling parameters. The scheduling parameters chosen for the hypersonic vehicle are altitude and Mach number. A breakdown of the parameter space can be seen in Fig. 4.

From Fig. 4, it can be seen that the continuous parameter space is discretized. Each discrete point in the parameter space has a linearized plant model associated with it. A linear controller is then designed such that the stable region for which the linear controller is valid overlaps with the next controller’s stable region. The LPV system is a class of linear systems with its state space matrices depending on a time-varying vector \( \rho(t) \in R^s \),

\[
\dot{x}(t) = A(\rho(t))x(t) + B_1(\rho(t))d(t) + B_2(\rho(t))u(t) \quad (30)
\]

\[
e(t) = C_1(\rho(t))x(t) + D_{11}(\rho(t))d(t) + D_{12}(\rho(t))u(t) \quad (31)
\]

\[
y(t) = x(t) \quad (32)
\]

It is assumed that the scheduling parameter \( \rho \) evolves continuously over time and its range is limited to a compact set \( \rho \in P \). In addition, its time derivative is often assumed to be bounded and satisfy \( \underline{\rho}_k < \dot{\rho}_k < \overline{\rho}_k \), \( k=1,2,\ldots,s \). Moreover, assume that

(A1) The matrix function pair \( (A(\rho), B_z(\rho)) \) is parameter-dependent stabilizable and detectable,

(A2) The matrices \( [C_2(\rho) \quad D_{2z}(\rho)] \) has full row rank for all \( \rho \in P \).

Similar to the open loop description, the LPV synthesis conditions also change with parameterization. For full state feedback, \( y=x \), consider the static state feedback control law in the form of
Using a parameter-dependent quadratic Lyapunov function $V(x) = x^T P(x) x$, the solution of $H_\infty$ synthesis problem of an LPV full state feedback problem is to find a continuously differentiable matrix function $R(\rho) > 0$ satisfies for any $\rho \in \mathcal{P}$

\[
\begin{bmatrix}
N_R(\rho) & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A(\rho)R(\rho) + R(\rho)A^T(\rho) - \sum_{i=1}^{s} \{x(i), \overline{v}_i\} \frac{\partial R}{\partial p_i} \\
C(\rho)R(\rho) & -\gamma I \\
B_1(\rho) & D_{11}(\rho) - \gamma I
\end{bmatrix} < 0
\]

where

\[
N_R(\rho) = \ker[B_2^T(\rho) \ D_2^T(\rho)]
\]

Consequently, the resulting LPV state feedback control gain will be:

\[
F(\rho) = (D_{12}(\rho)D_{12}(\rho))^{-1} \cdot \left[\gamma B_2^T(\rho)R(\rho) + D_{21}(\rho)C_1(\rho)\right]
\]

Now that the equations of motion and the LPV synthesis condition have been established, it is possible to find a solution to a given problem using LMI optimization techniques. It should be noted here that the synthesis conditions involve an infinite number of constraints. To simplify this problem, the parameters will be discretized. This means that a finite number of points for each parameter in the system will be chosen to represent the system as a whole. It is important to realize that each of these discretized parameter points will represent a linear model of the system. These gridding points will be chosen in a way such that the discrete points are close enough that the range for which each linearized region is valid overlaps with another linearized gridding point. This can be seen in Fig. 4. Moreover, $R(\rho)$ must also be parameterized using a finite number of basis functions as,

\[
R(\rho) = \sum_{i=1}^{N_f} f_i(\rho) R_i
\]

where $f_i(\rho), i = 1, 2, \ldots N_f$ is a user specified basis function. Once this parameterization of the system has taken place, the synthesis problem can now be solved.

IV. LPV Control Design and Nonlinear Simulation for HSV

The previous sections have discussed equations of motion and the LPV control design techniques. This section will discuss the design of LPV controller to a specific hypersonic vehicle. It will include a discussion of both actuator dynamics and digital switching techniques. The velocity tracking case will be considered in this study.

For this vehicle, the scheduling parameters chosen were altitude and Mach number. For a given altitude and Mach number, a linearized plant was created. The state space representation of this linearized plant is given by,

\[
\dot{x}_p = A_p x_p + B_p u
\]
where $C_p$ is just the identity since the system is using full state feedback. The plant state vector $x$ is defined as:

$$x_p = [V \ a \ Q \ h \ \theta \ \eta, \ \eta_1, \ \eta_2, \ \eta_3] ^T$$

This state vector describes the plant dynamics, but in this study it is also necessary to model a description of the actuator dynamics as well. This will help to capture actuator dynamic behavior for nonlinear simulation. The actuator dynamics were chosen so that they behave as a low pass filters in the system.

$$A_d = \begin{bmatrix}-20 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

Along with the plant dynamics, it is also necessary to add a state to the system to describe the tracking error. This state is defined as the integration of the tracking error between the reference velocity and the actual vehicle velocity.

The augmented plant dynamics take the following state space form:

$$H_p = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{aug} = \begin{bmatrix}A_p & B_p & 0 \\ 0 & A_d & 0 \\ -H_p & 0 & 0 \end{bmatrix}$$

$$B_{1aug} = \begin{bmatrix}0 \\ I \end{bmatrix}$$

$$B_{2aug} = \begin{bmatrix}0 \\ B_d \end{bmatrix}$$

$$C_{1aug} = w \cdot \begin{bmatrix}-H_p & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_{11aug} = w \cdot \begin{bmatrix}I \\ 0 \end{bmatrix}$$
where \( w \) in equations (47-49) is a weight designed to penalize the output of the system. The new state vector \( x \) becomes:

\[
x = [x_p \ \delta_e \ \delta_r \ \phi \ \delta_{diff} \ \int e]^T
\]  

The state vector in Eq. (50) shows that the four actuator states are now a part of the plant dynamics as is the integration of the error. Now that a linearized plant model has been accurately defined, a controller is synthesized for each discrete plant such that the synthesis condition in Eq. (37) is satisfied. \( H_\infty \) synthesis will optimize the system for disturbance rejection.

Now that the set of linear systems and controllers has been defined, it is necessary to design an algorithm for the switching between the different controllers. The set of linear controllers will be simulated with the nonlinear plant dynamics, so the controllers will need to be changed as the system moves through the scheduling parameter space. For this hypersonic vehicle, the gridding points were placed close enough together such that the linearized controllable region for a single point enveloped the linearized controllers next to it. This can be seen more clearly in Fig. 5. The switching algorithm used here calculates the Mach number from the velocity and altitude. It then uses the altitude and Mach number to find the nearest controller. This controller will be used until it passes the switching threshold. The switching threshold for this particular controller was set such that the system does not switch from one controller to the next until the system has reached the next controller. This was chosen instead of the midway point between the controllers to prevent high frequency switches between controllers being used by the system. Fig. 6 shows the switching threshold used for the system.

It was discovered that when switching took place in the system, there were times when the system went into flight conditions that were not physically possible. This was caused mainly by the nature of the switching controllers itself. When the system commands a controller switching, the nominal trim conditions for the system were changed. This caused a discontinuity in the control effort which lead to either detached shock waves or problems with subsonic flow in the scramjet depending upon the flight conditions. It is possible that the abrupt change in control effort lead to unsuitable flight conditions. To solve this problem, a least squares optimization was run to find a suitable value of \( \int e \) that will minimizes \( \|u\|_\infty \). Since \( \int e \) is a user defined reference variable and not a physical state of the system, it is possible to set this value to whatever the user desires in this study, it was set in order to minimize the amount of control effort needed at the

\[
D_{12au} = w \cdot [0]
\]  

Figure 5. Controller Region Layout.

Figure 6. Switching Threshold.
switching condition. This helped to minimize the amount of discontinuity seen in the control effort when a switch took place, and thus kept the system from reaching undesirable flight conditions that all other times it was allowed to run freely. This problem is very similar to the integral windup problem seen in PID controllers.

The hypersonic vehicle controller was synthesized for a Mach number that ranged from 7 to 9, and on an altitude range from 80,000 feet to 90,000 feet. Along each of these scheduling parameters, 7 evenly spaced points were chosen such that 49 total linear controllers were solved over the parameter space. The system was observed with a ramp input function for the velocity tracking problem. Fig. 7 shows the velocity command signal and the response of the HSV vehicle. The nonlinear system follows quite well to the reference signal for this case. It should be noted that there is a small bump up in the velocity at about 21 seconds. This is where the controller is switching from the first controller to the second controller. It is a relatively small change to the velocity tracking profile overall, and the system does continue to converge to the command signal.

Figure 8 shows the angle of attack for the system. The maximum angle of attack is roughly .078 radians and the minimum is about 0 radians. This is an acceptable range for a hypersonic vehicle.

Figure 9 shows the altitude of the hypersonic vehicle. It is interesting to see that the vehicle continuously loses altitude over the course of this simulation. There is no tracking effort spent dictating the altitude of the vehicle. As a result, it can be seen that in order to achieve the desired velocity with the least amount of control effort spent, the optimal path for the altitude is a steady decrease. This means that it is most efficient for the HSV to trade altitude for airspeed. This is acceptable, and is sensible given the constraints of the problem. It should be noted that the altitude has saturated the lower bound of the system by the end of the run. Though the altitude is still within the controllable region, it would be necessary to add an additional regulation or tracking state to the altitude if the HSV is to remain inside the controllable region while tracking the commanded velocity.

Figure 10 shows the angle of the elevator. This control effort has an upper saturation limit of \( \frac{\pi}{6} \) radians and a lower limit of \( -\frac{\pi}{12} \) radians. The switching moments can be seen clearly from this control input. A certain amount of settling time is needed after the switch to reduce the high frequency oscillations in the control effort. These oscillations are caused by the systems new trim conditions which are achieved at each new controller. The range and frequency of the elevator angle is within the acceptable limits for the hypersonic vehicle.
Figure 11 shows the angle of the canard. The upper saturation limit of the canard is $\frac{\pi}{9}$ radians and a lower saturation limit of $-\frac{\pi}{9}$ radians. The response of the canard is very similar to that of the elevator. There is a high frequency response component when the controller switching takes place. The velocity still tracks the reference command even though the canard has these high frequency components. The range and frequency of the canard remain within the acceptable range.

Figure 12 shows the fuel equivalence ratio (FER). The FER describes the ratio of the fuel to air as compared with the stoichiometric fuel to air ratio. The FER has an upper saturation limit set at .77 and a lower saturation at .1. This control effort essentially controls how rich or lean the fuel to air mixture is, and subsequently is directly related to the amount of thrust that the scramjet can produce. It can be seen that the FER saturates at the upper bound at about 22 seconds, and again at about 36 seconds. This is most likely due to a transient period in the system that occurs during the switching conditions. The velocity still tracks the command signal even with this saturation taking place; therefore this level of saturation is acceptable.
Figure 13 shows the diffuser area ratio. The diffuser area ratio has an upper saturation limit of 1 and a lower saturation limit of 0. It can be seen from Fig. 13 that the diffuser area ratio saturates at the upper bound initially, then it fluctuates between .78 and .98 roughly. The velocity continues to track the reference command even with this initially saturated actuator. The saturation on this control effort is acceptable.

![Diffuser Area Ratio Control Effort for Flexible Body](image)

**Figure 13. Diffuser Area Ratio.**

V. Conclusions

The outcome of this study is a LPV controller design method for a flexible hypersonic vehicle flying through a large envelope of flight conditions. The results from the simulation show that the actuation of the control efforts is feasible, and that the velocity of the vehicle does in fact track the reference velocity well. This study also shows some difficulty of implementing LPV controller for HSV when a switch between controllers is needed, but a suitable solution has been given for this problem. Additionally, the angle of attack was shown to be within a realistic range for the vehicle. The altitude response was realistic, but not desirable. There was no weight given to the tracking of the altitude in the system, so the response is not surprising, but it would be preferred to see the altitude holding a relatively constant height as opposed to the drastic change in height that is seen in this study.

Future work could include giving some sort of weighted tracking function to track both velocity and altitude. It would also be beneficial to look at doing an output feedback case for this vehicle in future research since measuring all of the states may not be feasible. By using output feedback, one could choose which states will be observed by the system.

Acknowledgments

Hunter Hughes thanks Michael Bolender and David Doman from the Air Force Research Lab for the contribution of their hypersonic vehicle code. Also, a special thanks to Mark Osborne for his significant contribution to this research. Additionally, Hunter Hughes thanks Scott Hays, Jason Bishop, Dave Burke, and Stearns Heinzen for their help with hypersonic vehicle dynamics.

References


